AN APPLICATION OF THE POPOV STABILITY CRITERION TO THE DESIGN OF NONLINEAR SYSTEMS

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THESIS

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ABSTRACT

A method for nonlinear system design, based on the generalized Popov stability criterion, is developed. It is shown that, although the design method is applicable for all values of q in the Popov theory, it is particularly useful when q is non-zero, since it greatly simplifies the design effort for this case. All designs are accomplished in conjunction with the Nyquist and modified Nyquist loci. Basic to the design procedure is the development and utilization of the modified frequency-response polar loci for lag and lead compensation networks. Three examples, one with digital simulation, are included to illustrate the procedure described.

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I. INTRODUCTION

The now well-known Popov criterion has provided a new approach to the idea of absolute stability in the dynamics of automatic control systems. As a result, the large amount of subsequent research has contributed much to the development of modern stability theory.

The importance of stability theory is quite evident when one realizes that all the current conventional design techniques for linear, time-invariant systems are directly or indirectly derived from stability theory. This is reflected in the fact that the basic engineering design approach is to determine how an unstable system can be altered to make it stable. When the system is nonlinear or time-varying, the design problem becomes much more complex. However, considerable progress has been made in this area in recent years.

This paper considers systems of the standard form of Figure 1. The linear element may contain time delays or distributed parameters. The nonlinear element may be time-varying and may contain hysteresis. The only restriction on the nonlinear element is that its input-output characteristic must lie within a sector bounded by two straight lines passing through the origin. There is no loss of generality if one line corresponds to the x-axis since this can be accomplished through a simple transformation. Hence:

$$0 \leq \frac{u(t)}{e(t)} \leq k \qquad \text{for all } t. \tag{1}$$

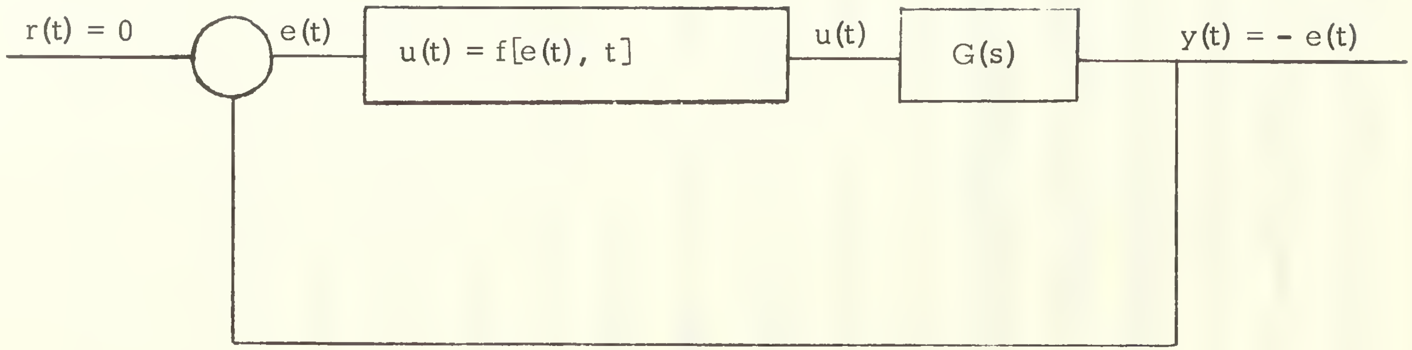

Fig. 1. The basic system considered in this paper.

This class of systems may be treated, in some cases, by phase-plane methods or by approximate methods such as the single or dual-input describing function. Some exact methods are also historically available.

The solution of the absolute stability problem was first formulated in 1944 by Lur'e and Postnikov [1] for the case when $u = f(e)$ is located completely in the first and third quadrants $(k = \infty)$. Also, in their paper, the utilization of a Lyapunov function of the type "a quadratic form plus an integral involving the nonlinearity" was first proposed. In further works, Lur'e developed a method which permits one to write down a finite system of quadratic equations directly from the given differential systems. He showed that if this system of equations, which he called resolving equations, has real roots, then this is a sufficient condition for the absolute stability of the system of Figure 1.

Yakubovitch [1] further developed this approach by showing that the system of resolving equations can be reduced to a single algebraic equation. He then proved that the existence of two real roots for this equation is a sufficient condition for absolute stability.

Most of the subsequent research was centered around the application of Lyapunov's method for determining system stability. However, up until this time, all the exact methods suffered from the disadvantage that even the solution of the simple problems became so formidable computationally that the methods were generally unusable as engineering tools.

Then, in 1959, Popov proposed an entirely different approach to the problem of absolute stability. He expressed his sufficient conditions in terms of the frequency response of the linear portion of the system, thus giving his criterion a very simple form that is convenient for graphical application. In fact, the Popov criterion is so powerful that all the results described above and connected with the Lyapunov function, consisting of a quadratic form plus an integral of the nonlinearity, are included in the criterion.

In the Popov method only the Nyquist diagram of the linear element and the inequality constraints of the nonlinear element are needed. Hence the criterion is readily applied to higher-order systems and those containing time delays or distributed parameters. Since it is basic to the design method developed in this paper a brief development of the Popov theorem follows. For a proof of the theorem, see [1].

In the most simple case, the problem of absolute stability of dynamic processes consists of finding when the equilibrium state of the general system

$$\begin{aligned} \dot{x} &= Ax + bu \\ u &= f(e) \\ e &= c'x \end{aligned} \tag{2}$$

where x is a vector, A is a square matrix, b is a column vector, c' is a row vector, and u is a scalar, is on the whole asymptotically stable in the Lyapunov sense. The formulation (2) is a system of differential equations differing from a linear homogeneous set with constant coefficients only by a nonlinear function whose argument is any one or a linear combination of the state variables.

A, b, and c are real, time-invariant matrices in which some of the components may be zero. The nonlinear characteristic, $f(e)$, is an arbitrary, single-valued, piecewise continuous real function, defined for all real values of e and satisfying the conditions

$$f(0) = 0 \tag{3}$$

$$0 \leq \frac{f(e)}{e} \leq k$$

which are interpreted geometrically in Figure 2.

The problem of establishing the absolute stability of the system (2) requires the following definitions. A system (2) such that all the roots of its linear portion characteristic equation lie in the left half-plane will be the so-called principal case of (2). However, a case when some of the roots of the characteristic equation are on the imaginary axis and the rest are in the left half-plane will be called a particular case of (2). In (2), assume that $f(e) = he$ with $0 \leq h \leq k$, for the principal case and with $0 < h \leq k$ for the particular cases. Then (2) becomes the linear system

$$\begin{aligned} \dot{x} &= Ax + hbe \\ c &= c's \end{aligned} \tag{4}$$

Now, Popov expressed his results in the frequency domain. In addition to the usual open-loop frequency response, $G(j\omega)$ (see Figure 1), he used a modified frequency response, $G^*(j\omega)$, which is defined as follows

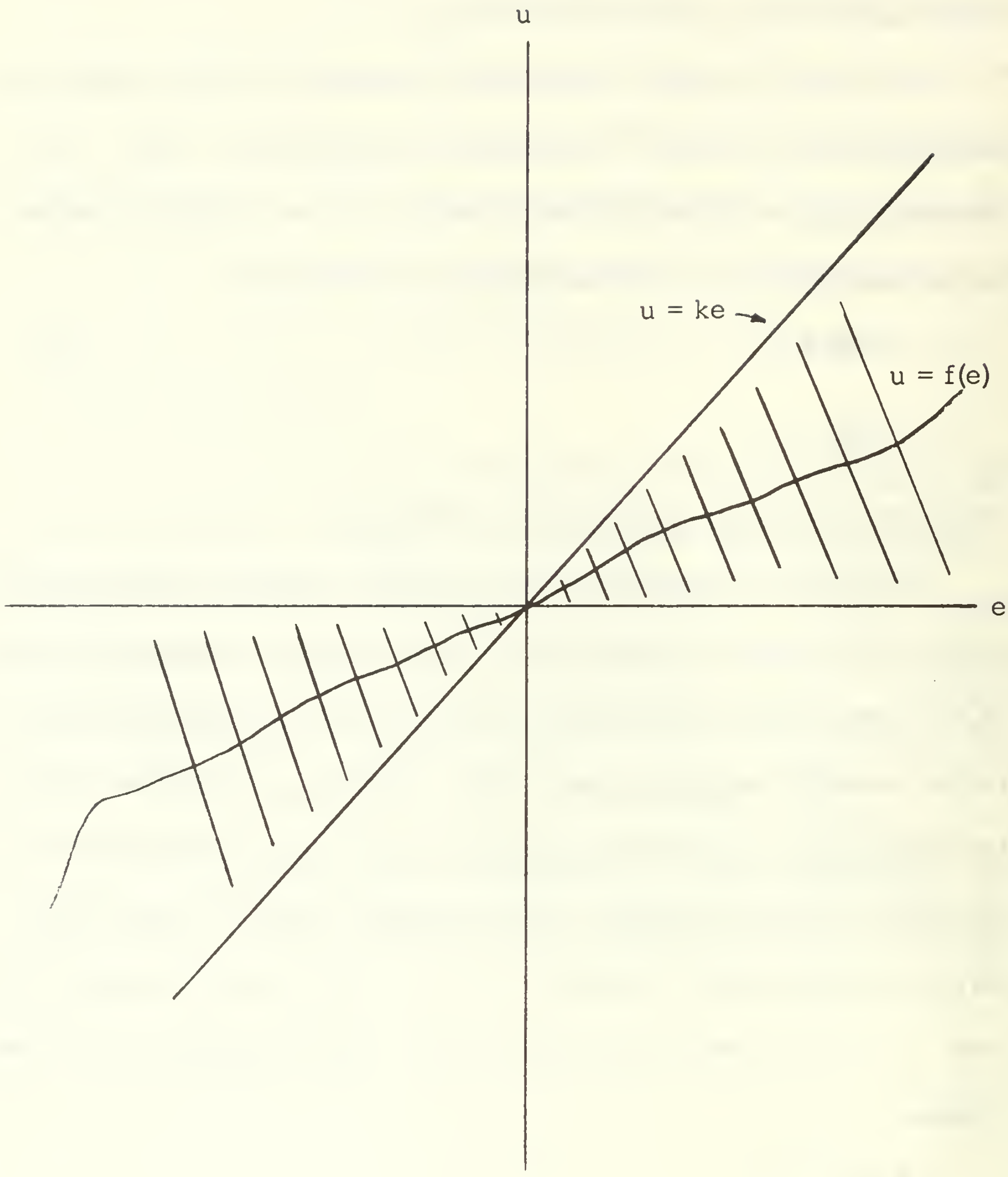

Fig. 2. The input-output characteristic of the nonlinear element.

$$\begin{aligned} \operatorname{Re} G^*(j\omega) &= \operatorname{Re} G(j\omega) = X \\ \operatorname{Im} G^*(j\omega) &= \omega \operatorname{Im} G(j\omega) = Y \,. \end{aligned} \tag{5}$$

From linear theory, for the system (4), obtained from the principal case of (2), to be stable for any h in the interval $0 \le h \le k$, it is necessary and sufficient that the locus of $G(j\omega)$, and hence also the locus of $G^*(j\omega)$, never intersect the real axis from negative infinity up to and including the point $(-1/k)$. Also for the system (4), corresponding to particular cases of (2), to be stable with $0 < h \le k$, it is necessary and sufficient that the locus of $G(j\omega)$ (or of $G^*(j\omega)$) never intersect the forbidden zone (the interval of the negative real axis from negative infinity to $-1/k$) and that the system (4) be stable-in-the-limit, that is, that the system (4) be stable for sufficiently small $h > 0$ (that is, $h = \epsilon > 0$ where ϵ is arbitrarily small).

Consider now the original problem of establishing the absolute stability of the system (2), that is, when $f(e)$ is an arbitrary (not necessarily linear) characteristic contained in the sector $(0, k)$ for the principal case, or in the sector (ϵ, k) for the particular cases. Additional conditions must be imposed on the frequency response in order to guarantee stability for the system (2), not only for linear characteristics $f(e) = he$, but for any nonlinear characteristics $f(e)$ contained in the given sector. Popov stated these additional requirements in the following famous theorem [1].

Theorem. For the system (2) to be absolutely stable in the sector $(0, k)$ for the principal case, and in the sector (ϵ, k) for the particular cases (where $\epsilon > 0$ is an arbitrarily small number), it is sufficient that there exist a finite real number q such that for all $\omega \geq 0$ the following inequality is satisfied

$$\mathrm{Re}(1 + jq\omega)G(j\omega) + 1/k > 0 \qquad (6)$$

and, additionally for the particular cases, that the conditions for stability-in-the-limit be satisfied.

In order to give the criterion a geometric interpretation let

$$G^*(j\omega) = X + jY \qquad (7)$$

Then

$$\mathrm{Re}(1 + jq\omega)G(j\omega) = \mathrm{Re}G(j\omega) - qw\mathrm{Im}G(j\omega) = X - qY.$$

Hence (6) can be written as

$$X - qY + 1/k > 0 \quad \text{for all } \omega \geq 0 \ . \qquad (9)$$

But the equation

$$X - qY + 1/k = 0 \qquad (10)$$

is the equation of a straight line (the Popov line) with slope 1/q which passes through the point -1/k on the real axis. Hence it is possible to give a geometric interpretation to the Popov theorem.

For the system (2) to be absolutely stable in the sector (0,k) for the principal case, or in the sector (ϵ , k) for the particular cases, it is sufficient that there exists in the $G^*(j\omega)$ plane a straight line, passing through the point -1/k on the real axis, such that the modified frequency response $G^*(j\omega)$ lies strictly to the right of it. Also, for the particular cases, the conditions for stability-in-the-limit must be satisfied. Some examples are shown in Figure (3).

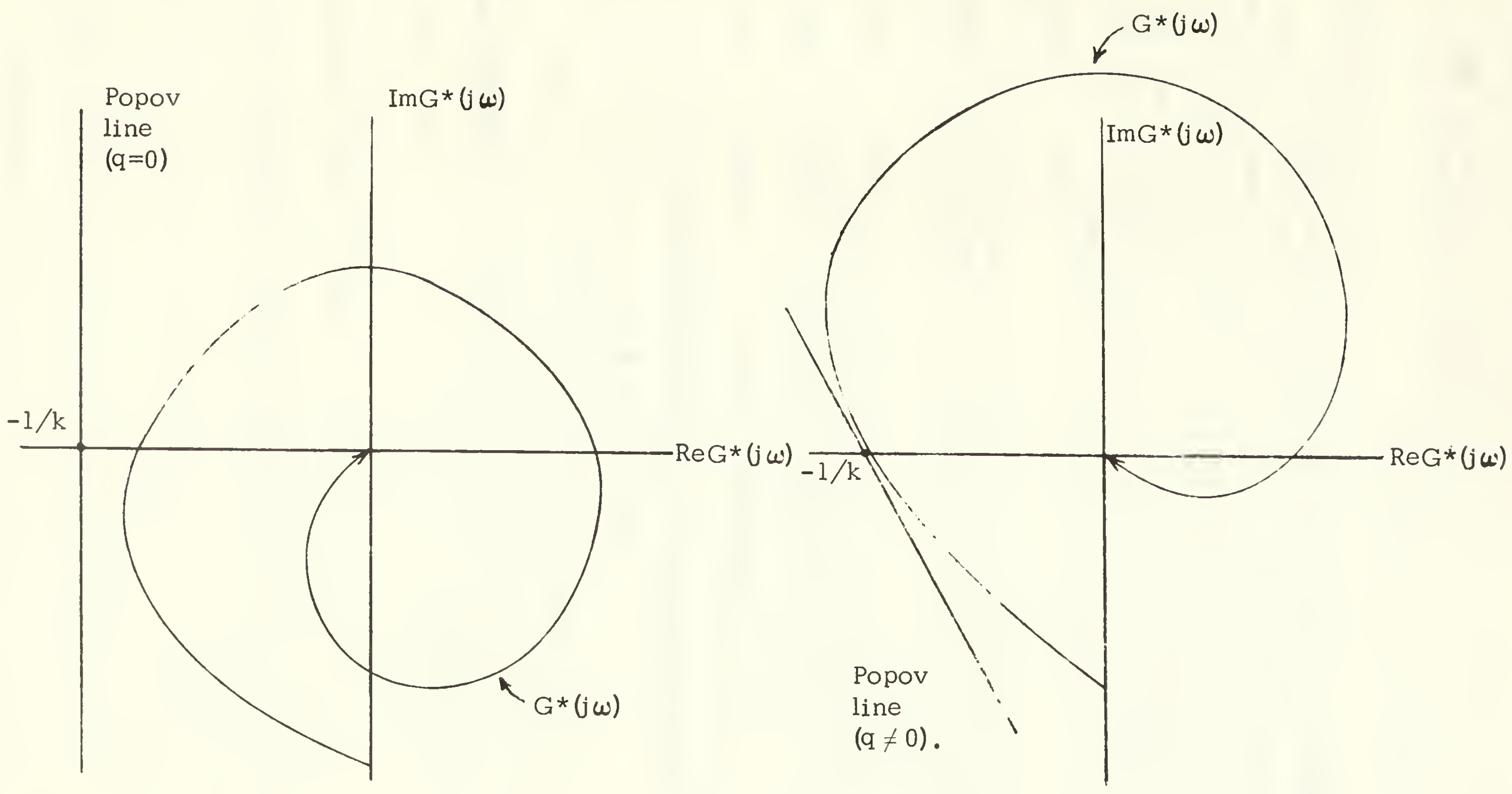

Fig. 3. Examples of the application of the Popov criterion.

Attempts to extend the Popov theorem have met with considerable success. Hsu and Meyer [2] have presented a generalized Popov theorem which includes the recent extensions.

Theorem. Consider the basic system of Figure 1 with the linear element being output stable. For the system to be both absolutely control and output asymptotic for $(u/e) \in (0/k)$, it is sufficient that there exists a real number q such that for all real $\omega \geq 0$ and an arbitrarily small $\delta > 0$, the following inequality is satisfied:

$$\operatorname{Re}\left((1 + j\omega q)G(j\omega) \right) + 1/k \geq \delta > 0 \qquad (11)$$

The restrictions on q and k depending on the nature of the nonlinear element, are:

1. for a single-valued, time-invariant element:

   if $0 < k < \infty$, then $-\infty < q < \infty$

   if $k = \infty$, then $0 \leq q < \infty$

2. for a nonlinearity with passive hysteresis:

   $0 < k < \infty$ and $-\infty < q \leq 0$

3. for a nonlinearity with active hysteresis:

   $0 < k \leq \infty$ and $0 \leq q < \infty$

4. for a general nonlinearity (time-varying, and possibly with hysteresis):

   $0 < k \leq \infty$ and $q = 0$

If $k = \infty$ in cases (1), (2), or (4) above, then an additional requirement must be met. For all t and for every value e_m, there must exist a finite value u_m such that

$$|u(t)| \leq u_m < \infty \text{ if } |e(t)| \leq e_m \qquad (12)$$

It can be noted immediately that the term "absolutely stable" has been replaced by "absolutely control and output asymptotic". This is mathematically the least restrictive practical stability definition. It simply means that the control signal or the output signal eventually go to zero.

A valuable engineering tool provided by this theorem is the trade-off available between the requirements on the linear element and those on the nonlinear element. For example, if the nonlinearity is single-valued and time invariant, a non-zero q may be chosen, thus providing a greater range for $G(j\omega)$.

The Popov criterion provides another valuable asset to the engineer. A measure of the rate of system damping is easily incorporated in the basic theorem. If the linear element of the system of Figure 1 is output stable of degree α, this simply means that, for $\alpha > 0$, the system output, in response to either an initial condition or an impulse, tends to zero faster than the function $e^{-\alpha t}$. For $\alpha < 0$, it means that the output multiplied by $e^{\alpha t}$ ultimately goes to zero.

Now it can be said that if the system of Figure 1 is control asymptotic of degree α, there exists a real number α such that for every set of initial conditions

$$\int_0^\infty (e^{\alpha t} u(t))^2 \, dt < \infty . \tag{13}$$

Also, it is output asymptotic of degree α if there exists a real number α such that for every set of initial conditions

$$\int_0^\infty (e^{\alpha t} e(t))^2 \, dt < \infty . \tag{14}$$

In order for the Popov inequality to reflect a measure of damping, the linear element must be output stable of degree α, and $G(j\omega-\alpha)$ must be substituted for $G(j\omega)$. Hence, for the system of Figure 1 to be control and output asymptotic of degree α, the following inequality must be satisfied:

$$\mathrm{Re}\ \left((1+jq\omega)G(j\omega-\alpha) \right) + 1/k \geq \delta > 0 \qquad (15)$$

Methods of system design based on the Popov criterion have been developed by several authors. See, for example, [2], [3], and [4]. These methods are all relatively straightforward and easily applied for the case $q = 0$, beacuse of the invariance of the Popov line with frequency on the Nyquist plot. However, for the case $q \neq 0$, the design of lag-lead compensating networks involves considerable design effort, much of it devoted to trial-and-error computation.

A great deal of this labor is eliminated in the new design method proposed here for the large group of systems in which design requirements dictate using the case $q \neq 0$. In order to retain the unparalleled utility of the polar-plot representation, the lag-lead compensating networks are designed on the $G^*(j\omega)$ plane.

II. DESIGN METHOD

A sample design objective is first illustrated in terms of a particular system. Then the general design method is developed.

Assume that the linear element of Figure 1 is output stable and has the Nyquist plot shown in Figure 4. Suppose also that the system design requires the nonlinear element to be a saturating amplifier with the input-output characteristic of Figure 5.

The generalized Popov criterion for a general nonlinearity ($q=0$) gives a maximum value of $k = 1.67$ for the upper bound on the Popov sector to guarantee control and output asymptoticity of the system. This is shown in Figure 4. However, since the particular nonlinearity of this system falls under Case 1, a non-zero q may be chosen to obtain a less conservative result. In order to do this, the simple transformation is made from the $G(j\omega)$ plane to the $G^*(j\omega)$ plane by multiplying the ordinates of the $G(j\omega)$ curve by their respective frequency values. The result is the modified frequency response, $G^*(j\omega)$. Assume that $G^*(j\omega)$ has the form shown in Figure 6. Drawing a tangent line to the $G^*(j\omega)$ curve (see Figure 6) at the intersection of the $G^*(j\omega)$ curve and the negative real axis gives a value for q of 0.1 and a value for k of 2.0.

The system nonlinearity (see Figure 5), however, requires a Popov sector with an upper bound of at least 9.0 in order to guarantee a stable system with the chosen amplifier. Hence the system must be compensated. Excessive compensation and a too conservative solution can be avoided by choosing a non-zero q. The design of lag or lead cascade

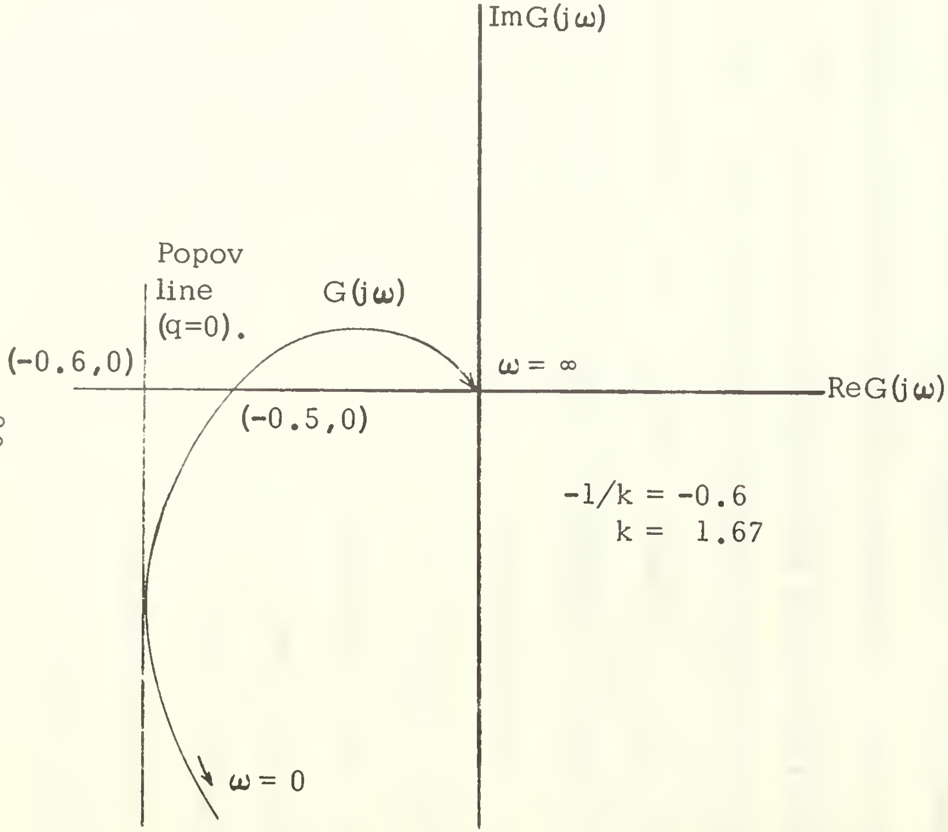

Fig. 4. Linear system polar plot.

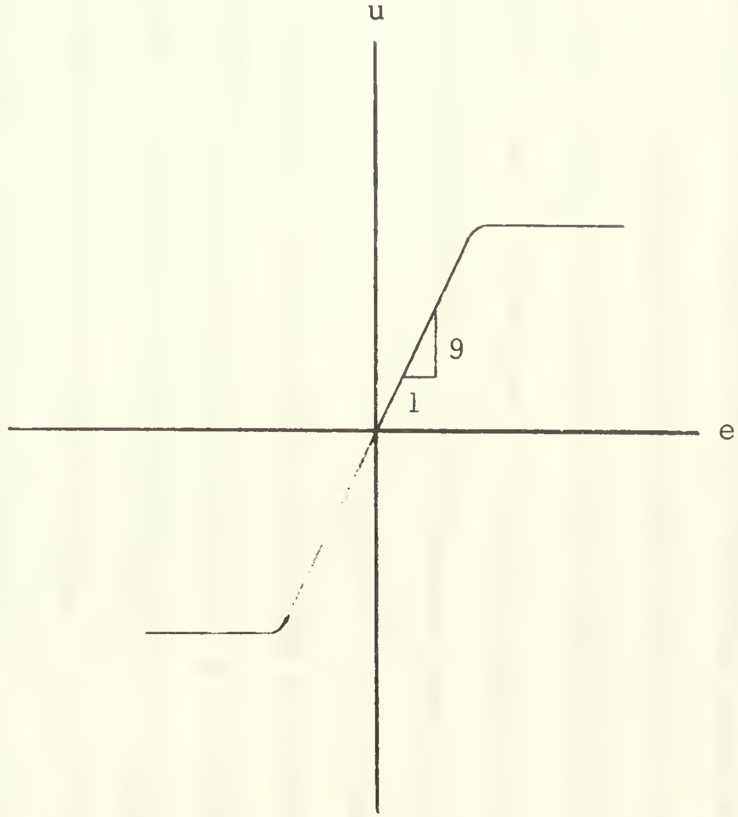

Fig. 5. Saturating amplifier input-output characteristic.

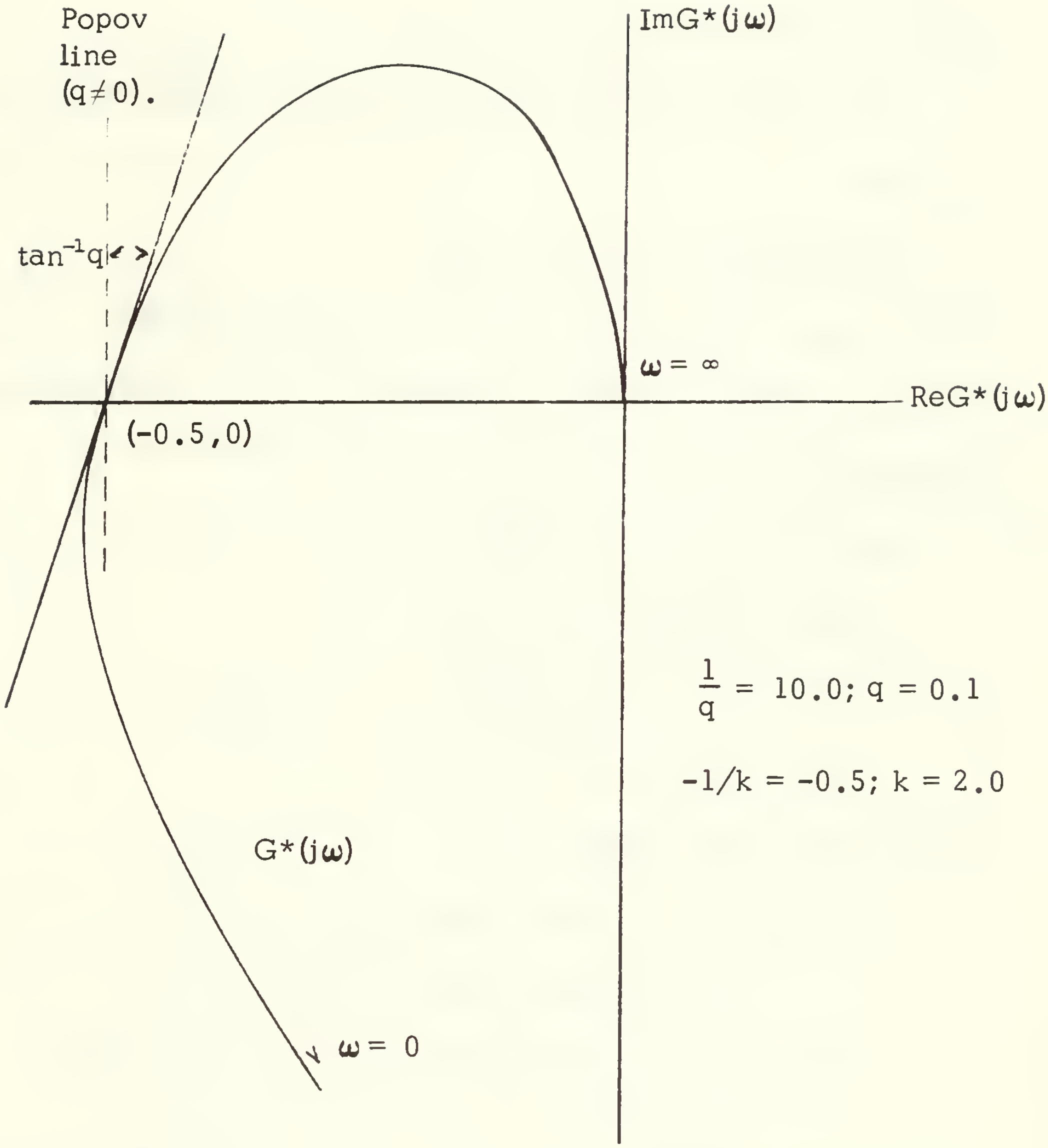

Fig. 6. Polar plot of the linear element modified frequency response.

compensation of the G* plane for systems of this type can be accomplished by the method described below with an appreciable decrease in labor over previous methods. The compensator is to be inserted in the output of the nonlinear element in the system of Figure 1. Before proceeding with the compensator design, several preliminary relationships which are instrumental to the design method are given.

One of the basic difficulties of designing in the $G^*(j\omega)$ domain is the fact that

$$[G(j\omega)G_c(j\omega)]^* \neq G^*(j\omega)G_c^*(j\omega) \tag{16}$$

where $G_c(j\omega)$ is the transfer function of the compensator. In order to show later how this difficulty may be minimized, the exact relationship is derived.

$$\begin{aligned} G(j\omega) &= \mathrm{Re}G(j\omega) + j\mathrm{Im}G(j\omega) \\ G_c(j\omega) &= \mathrm{Re}G_c(j\omega) + j\mathrm{Im}G_c(j\omega) \\ G^*(j\omega) &= \mathrm{Re}G(j\omega) + j\omega\mathrm{Im}G(j\omega) \\ G_c^*(j\omega) &= \mathrm{Re}G_c(j\omega) + j\omega\mathrm{Im}G_c(j\omega) \end{aligned} \tag{17}$$

Multiplying $G(j\omega)G_c(j\omega)$,

$$\begin{aligned} G(j\omega)G_c(j\omega) = {} & \mathrm{Re}G(j\omega)\mathrm{Re}G_c(j\omega) - \mathrm{Im}G(j\omega)\mathrm{Im}G_c(j\omega) \\ & + j[\mathrm{Re}G(j\omega)\mathrm{Im}G_c(j\omega) + \mathrm{Re}G_c(j\omega)\mathrm{Im}G(j\omega)] \end{aligned} \tag{18}$$

$$\begin{aligned} \therefore\ [G(j\omega)G_c(j\omega)]^* = {} & \mathrm{Re}G(j\omega)\mathrm{Re}G_c(j\omega) - \mathrm{Im}G(j\omega)\mathrm{Im}G_c(j\omega) \\ & + j\omega[\mathrm{Re}G(j\omega)\mathrm{Im}G_c(j\omega) + \mathrm{Re}G_c(j\omega)\mathrm{Im}G(j\omega)]\,. \end{aligned} \tag{19}$$

Multiplying $G^*(j\omega)G_c^*(j\omega)$,

$$\begin{aligned} G^*(j\omega)G_c^*(j\omega) = {} & \mathrm{Re}G(j\omega)\mathrm{Re}G_c(j\omega) - \omega^2\mathrm{Im}G(j\omega)\mathrm{Im}G_c(j\omega) \\ & + j\omega[\mathrm{Re}G(j\omega)\mathrm{Im}G_c(j\omega) + \mathrm{Re}G_c(j\omega)\mathrm{Im}G(j\omega)]\,. \end{aligned} \tag{20}$$

Comparing (19) and (20),

$$[G(j\omega)G_c(j\omega)]^* = G^*(j\omega)G_c^*(j\omega) + (\omega^2-1)\mathrm{Im}G(j\omega)\mathrm{Im}G_c(j\omega) \quad . \qquad (21)$$

The frequency-response plots of the compensation networks are essential to the design procedure. The transfer function of a lag compensation network is given by

$$G_c(s) = \frac{1 + s\tau_2}{1 + s\tau_1} \quad \text{where} \quad \tau_1 > \tau_2 \qquad (22)$$

and that of a lead network is given by

$$G_c(s) = \frac{1 + s\tau_1}{1 + s\tau_2} \quad \text{where} \quad \tau_1 > \tau_2 \ . \qquad (23)$$

The Nyquist plots of the two networks are shown in Figure 7.

The geometric configuration of the modified frequency responses, $G_c^*(j\omega)$, is now derived for the lag filter only. The lead filter gives the same result only with τ_1 and τ_2 reversed. For the lag filter:

$$G_c^*(j\omega) = \frac{1 + \omega^2\tau_1\tau_2}{1 + \omega^2\tau_1^2} + j\ \frac{\omega^2(\tau_2 - \tau_1)}{1 + \omega^2\tau_1^2} \ . \qquad (24)$$

$$\therefore\ X = \frac{1 + \omega^2\tau_1\tau_2}{1 + \omega^2\tau_1^2} \ ; \quad Y = \frac{\omega^2(\tau_2 - \tau_1)}{1 + \omega^2\tau_1^2} \ . \qquad (25)$$

Solving for ω^2 in the X-expression:

$$(1 + \omega^2\tau_1^2)X = 1 + \omega^2\tau_1\tau_2$$

$$X + \omega^2\tau_1^2 X = 1 + \omega^2\tau_1\tau_2$$

$$\omega^2\tau_1^2 X - \omega^2\tau_1\tau_2 = 1 - X$$

$$\therefore\ \omega^2 = \frac{1 - X}{\tau_1^2 X - \tau_1\tau_2} \qquad (26)$$

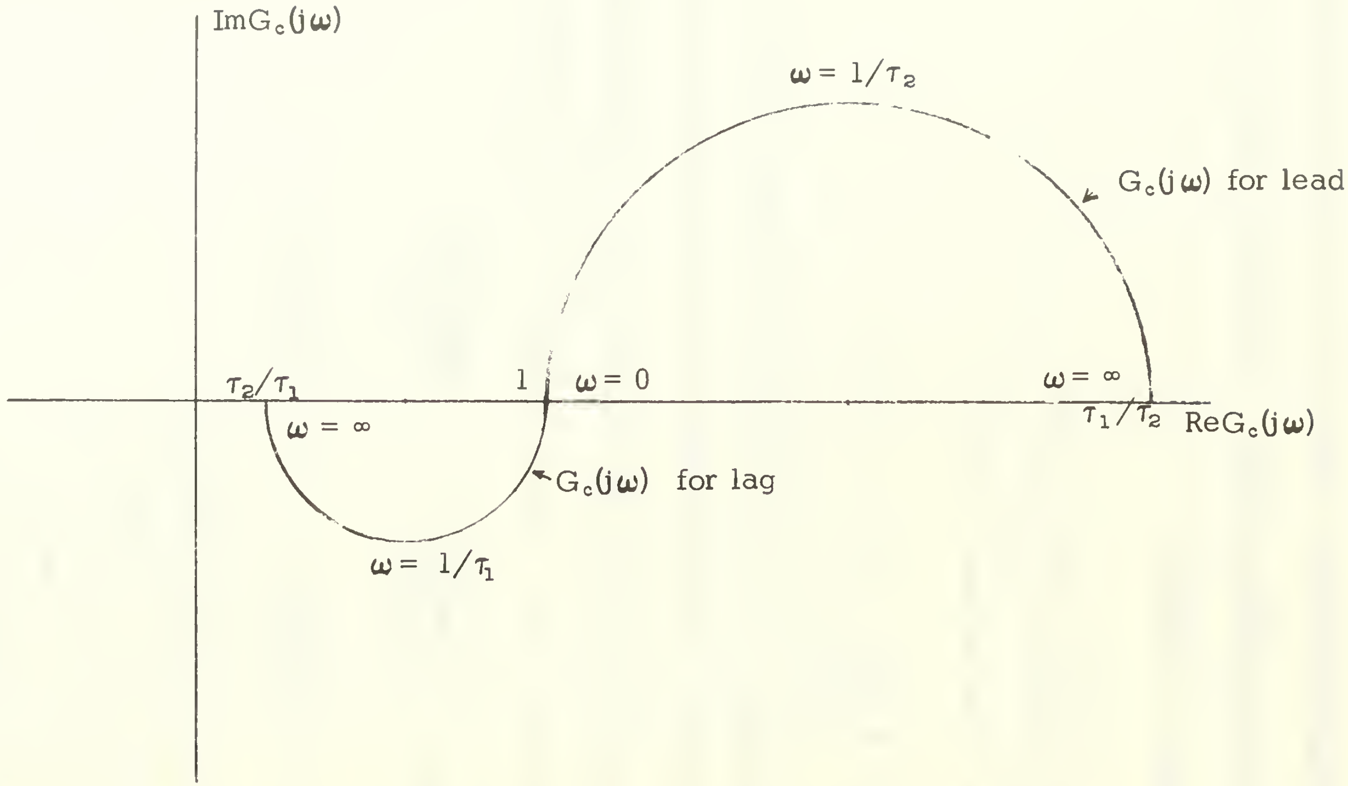

Fig. 7. Lag and lead compensation network polar plots.

Substituting into the Y-expression:

$$Y = \frac{\dfrac{1 - X}{\tau_1^2 X - \tau_1 \tau_2} \cdot (\tau_2 - \tau_1)}{1 + \dfrac{1 - X}{\tau_1^2 X - \tau_1 \tau_2} \cdot \tau_1^2}$$

$$= \frac{\dfrac{(1-X)\ (\tau_2 - \tau_1)}{\tau_1^2 X - \tau_1 \tau_2}}{\dfrac{\tau_1^2 X - \tau_1 \tau_2 + (1-X)\tau_1^2}{\tau_1^2 X - \tau_1 \tau_2}}$$

$$= \frac{(1 - X)\ (\tau_2 - \tau_1)}{\tau_1^2 X - \tau_1 \tau_2 - \tau_1^2 X + \tau_1^2}$$

$$= \frac{(1-X)\ (\tau_2 - \tau_1)}{\tau_1\ (\tau_1 - \tau_2)}$$

$$= \frac{(X-1)\ (\tau_2 - \tau_1)}{\tau_1\ (\tau_2 - \tau_1)} \ .$$

Therefore,

$$Y = 1/\tau_1\ (X-1)\ . \tag{27}$$

Similarly for the lead filter,

$$Y = 1/\tau_2\ (X-1)\ , \tag{28}$$

which is, of course, the equation of a straight line. This very convenient result is instrumental to the design method. In order to compute the end points of the line (24) is simply evaluated at $\omega = 0$ and $\omega = \infty$. The results for both filters are shown in Figure 8. The frequencies $\omega = 1/\tau_1$ and $\omega = 1/\tau_2$ occur at the geometric means of the respective polar plots.

The ideas developed so far can now be applied to the system with the modified frequency response shown in Figure 6. Suppose we decide to compensate this system with a lag compensator to force the

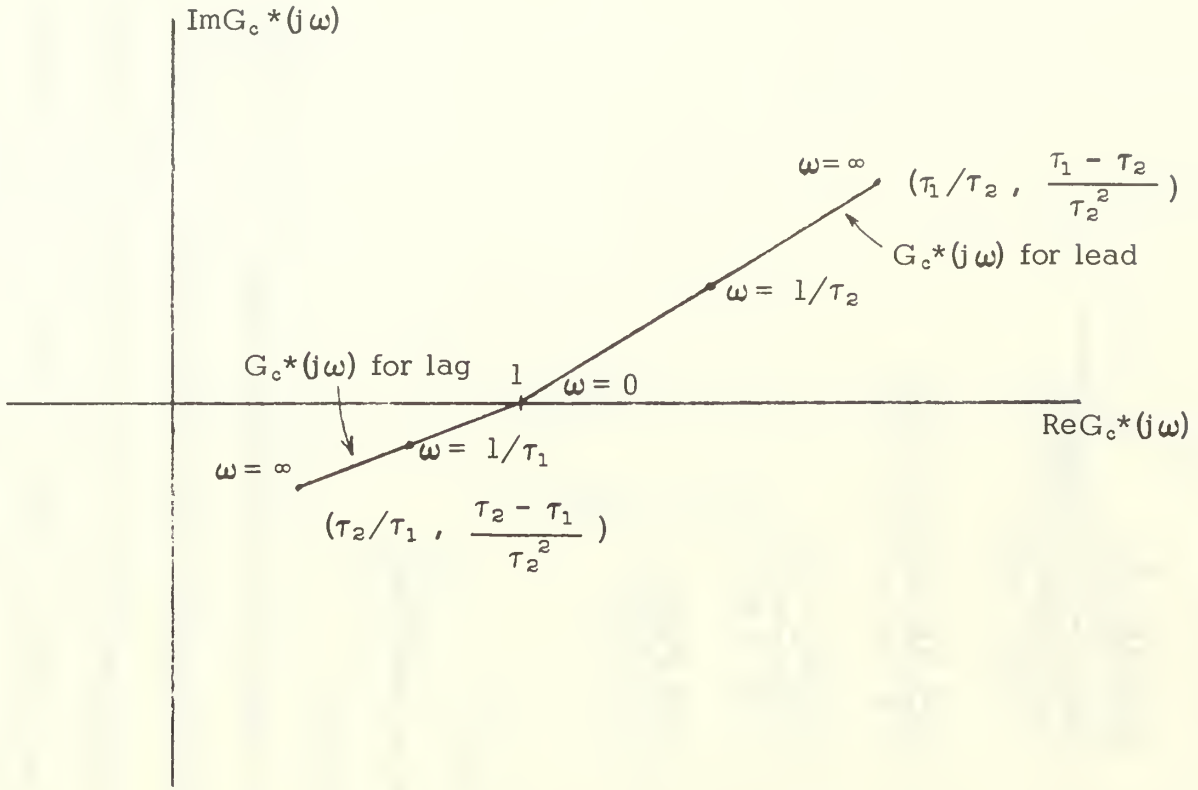

Fig. 8. Modified frequency response polar plots for lag and lead compensation networks.

compensated locus, $[G(j\omega)G_c(j\omega)]^*$, to intersect the negative real axis at a point ≥ -0.1 and at a certain crossover frequency, ω_c. This would give a Popov sector of at least $(0, 10)$ which would guarantee a stable system with the required amplifier, provided that a Popov line of non-zero q can be drawn through the point $(-0.1, 0)$.

The procedure to accomplish this alteration of the $G^*(j\omega)$ locus is first outlined briefly, then in more detail. The desired negative real axis crossover point for the compensated locus is designated as P_o. Then the $1/G^*(j\omega)$ and the $G_c^*(j\omega)/P_o$ loci are constructed so as to intersect at the point P_1 where the value of the $1/G^*(j\omega)$ locus corresponds to the chosen crossover frequency, w_c, and the parameters, τ_1 and τ_2, are then computed so that the frequencies of the two curves are equal at the intersection point. This construction designs the compensator since it guarantees that

$$1/G^*(j\omega) = G_c^*(j\omega)/P_o \quad \text{at} \quad \omega = \omega_c$$

or

$$G^*(j\omega)G_c^*(j\omega) = P_o \quad \text{at} \quad \omega = \omega_c$$

where $G^*(j\omega)G_c^*(j\omega)$ and $[G_c(j\omega)G(j\omega)]^*$ are related by (21). This relationship is discussed in detail later.

To show how easily this is accomplished in practice, the example of this section is used. Let $P_o = -0.1$ and $\omega_c = 1.0$. The first step would be to construct the $1/G^*(j\omega)$ curve. However, this is unnecessary since only one point of the $1/G^*(j\omega)$ curve is actually used, the one corresponding to $\omega_c = 1.0$. The coordinates of this point, P_1, are easily computed and plotted by inverting the complex number read off the $G^*(j\omega)$ locus for $\omega_c = 1.0$.

The next step is to construct the $G_c^*(j\omega)/P_o$ plot by simply rotating the $G_c^*(j\omega)$ plot by 180° and placing the $\omega = 0$ point of the compensator at $1/P_o$. The slope of this straight-line plot is adjusted so that the line passes through the point P_1. Thus P_1 corresponds to the intersection of the $1/G^*(\omega)$ and the $G_c^*(j\omega)/P_o$ loci, and the frequency at this point must be the same, i.e., $\omega = \omega_c$, on both curves. Refer to Figure 9.

Equation (27) gives the equation of the lag-compensator plot. By comparing this with the standard point-slope form of a straight line, it can be seen that the slope is $1/\tau_1$. Hence τ_1 can be determined by simply computing the slope of the line graphically. Then τ_2 is calculated from either part of (25) using $\omega = \omega_c$. The coordinates, X and Y, are simply given by the real and imaginary parts of P_1/P_o. This is a very easy calculation since the value of the squared parameter (in this case, τ_1) is always substituted numerically, whether the filter is lag or lead. Using $\omega = \omega_c$ in the calculation of τ_2 guarantees that the frequency of the $G_c^*(j\omega)/P_o$ function at P_1 is the same as that of $1/G^*(j\omega)$ at P_1.

To avoid a sign error when computing P_1/P_o, it should be noted that X and Y are always positive for a lead compensator and X is positive and Y is negative for a lag compensator. It should also be noted that there are slope limitations on the straight-line compensator plots. Infinite

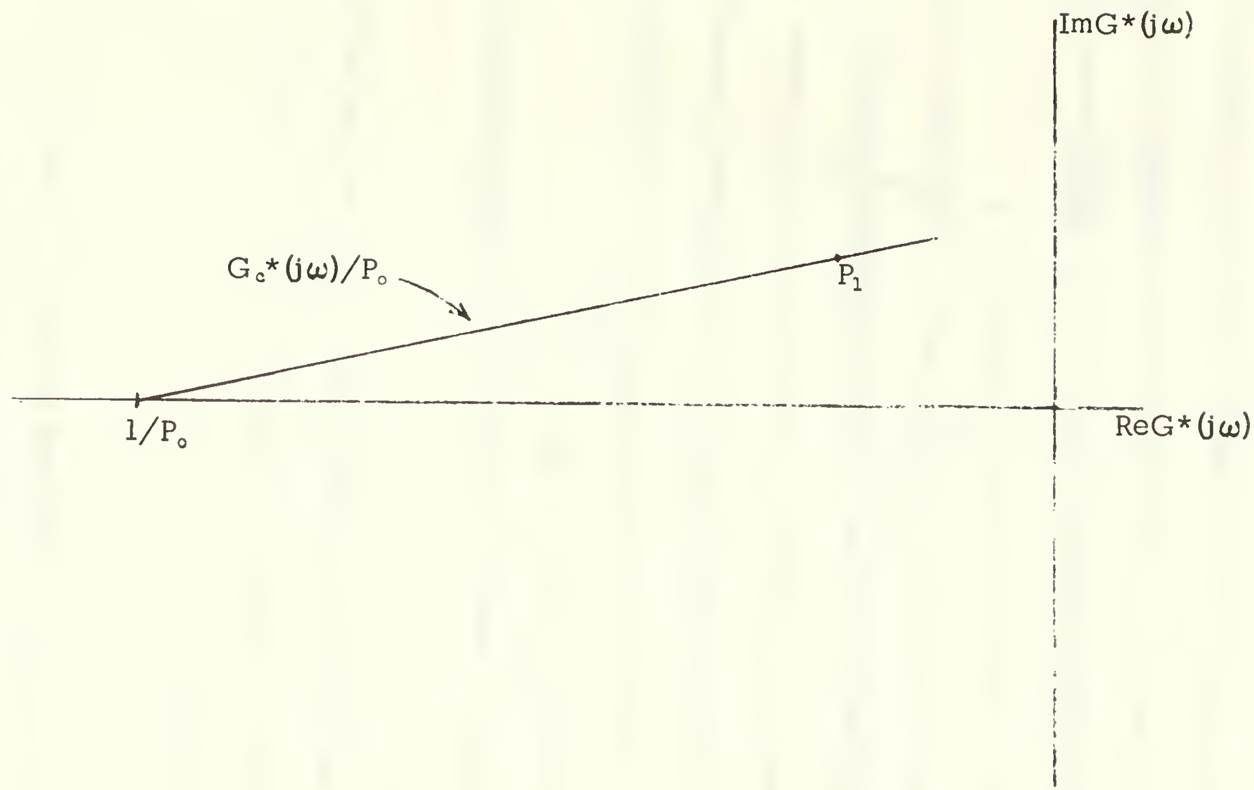

Fig. 9. Lag network compensation design.

slope corresponds to the pole and zero which cancel each other while zero slope corresponds to infinite pole-zero separation.

The frequency response curve of interest in the system design is the $[G(j\omega)G_c(j\omega)]^*$ curve. The plot given by the compensation method explained above gives the $G^*(j\omega)G_c^*(j\omega)$ curve. Equation (21) relates these two quantities. Note that since ω_c was chosen to equal 1.0 in the above description, the two curves cross the negative real axis at the same point. This is not the case in general.

Assume that the two resulting curves are as shown in Figure 10. Here a non-zero q line can be drawn through the point (-0.1,0) and the system design is completed. In an actual design problem there may be other performance specifications, such as minimum bandwidth, which must be met before the design is complete. These additional requirements will influence the choice of ω_c and P_o

In order to generalize the method to any ω_c, (21) is repeated for reference.

$$[G(j\omega)G_c(j\omega)]^* = G^*(j\omega)G_c^*(j\omega) + (\omega^2-1)\mathrm{Im}G(j\omega)\mathrm{Im}G_c(j\omega). \qquad (21)$$

For simplicity, let $(\omega^2-1)\mathrm{Im}G(j\omega)\mathrm{Im}G_c(j\omega) \equiv T$. It can be noted immediately that

$$T = 0 \text{ when } \omega = \begin{cases} 1 \\ 0 \text{ and } \infty \text{ since } \mathrm{Im}G_c(j\omega) = 0 \\ \text{value for which } \mathrm{Im}G(j\omega) = 0 . \end{cases} \qquad (29)$$

For the compensated system to be worsened by the effect of T, (i.e. the $[G(j\omega)G_c(j\omega)]^*$ curve be more negative than the $G^*(j\omega)G_c^*(j\omega)$ curve

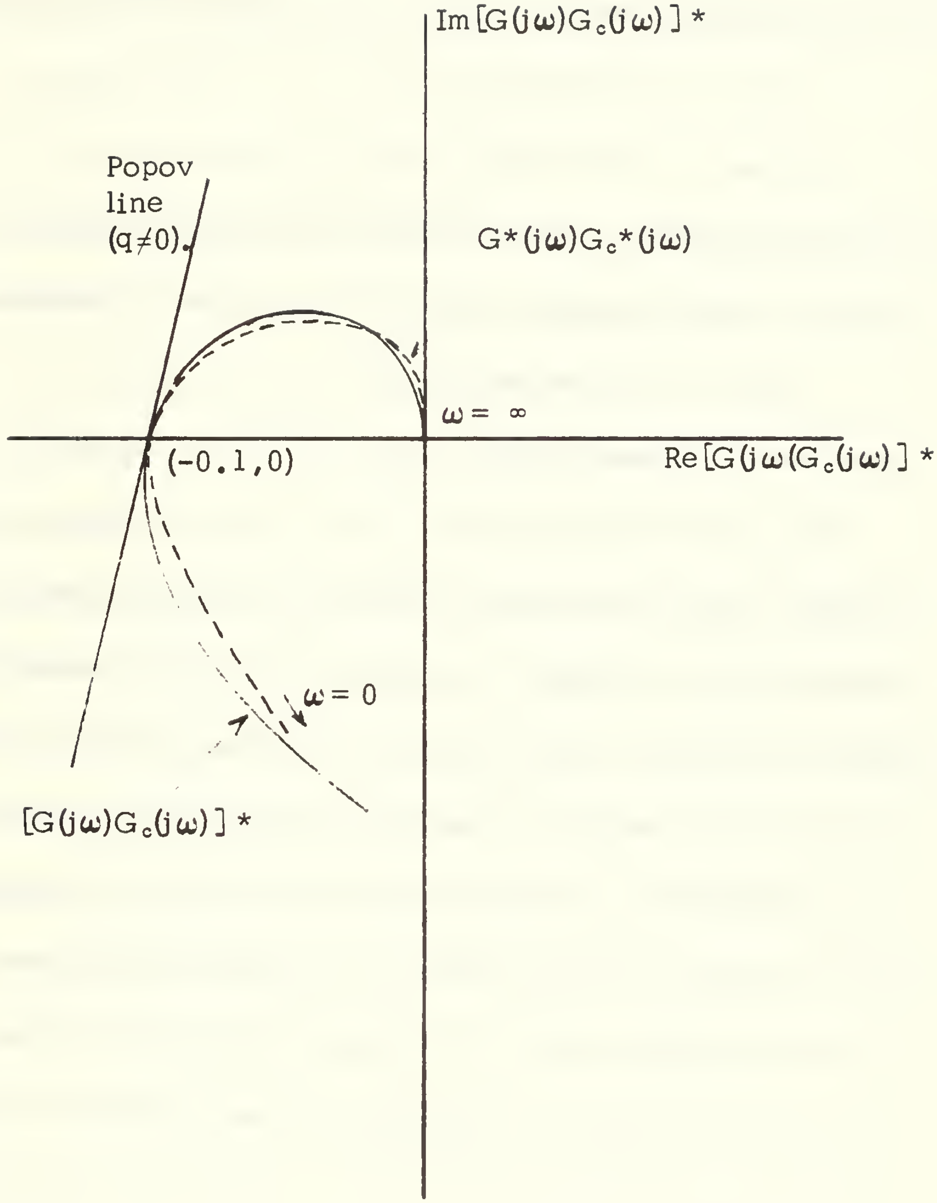

Fig. 10. Modified frequency-response locus for compensated system.

at the negative real axis crossover), the term T must be negative for $\omega = \omega_c$.

For a lag network, since $\mathrm{Im}G_c(j\omega)$ equals zero or a negative value for all ω, this can occur only under the following two conditions:

$$\text{if } \omega_c < 1 \text{ and } \mathrm{Im}G(j\omega_c) < 0$$

$$\text{or if } \omega_c > 1 \text{ and } \mathrm{Im}G(j\omega_c) > 0 \; . \tag{30}$$

For a lead network, since $\mathrm{Im}G_c(j\omega)$ equals zero or a positive value for all ω, T can be negative only under the following conditions:

$$\text{if } \omega_c < 1 \text{ and } \mathrm{Im}G(j\omega_c) > 0$$

$$\text{or if } \omega_c > 1 \text{ and } \mathrm{Im}G(j\omega_c) < 0 \; . \tag{31}$$

Using the methods outlined above, a rapid design procedure can now be stated. Since the subject of compensation is treated in most control system design textbooks, no attempt will be made in this paper to summarize the various helpful hints given by other authors for compensation design of linear systems using lag and lead networks. Since the design method outlined here involves reshaping a linear-system frequency-response curve, it is assumed that the designer has a feel for the steady-state and transient effects of compensation on linear system performance and is cognizant of the advantages and disadvantages of lag and lead compensators. See, for example, (5). With this in mind, the design steps may be listed as follows:

1. Plot the polar curves for $G(j\omega)$ and $G^*(j\omega)$ as ω varies from 0 to ∞. The $G^*(j\omega)$ plot is easily made by multiplying the ordinates of the $G(j\omega)$ locus by their respective frequencies.

2. From the required design specifications and system components choose the type of compensation (lag or lead) and W_c .

3. Use (30) or (31) and the design specifications for the initial choice of P_o .

4. Design the compensator.

5. Sketch the $G^*(j\omega)G_c^*(j\omega)$ locus by evaluating the compensator of Step 4 for a minimum number of frequencies. Sketch, if necessary, the $[G(j\omega)G_c(j\omega)]^*$ locus using (21).

6. Evaluate the design in view of the specified performance criteria.

If multiple sections are required the process is simply repeated using the $[G(j\omega)G_c(j\omega)]^*$ curve in Step 1. Since the design method is virtually identical for lead and lag compensators, a detailed discussion of lead network design is left to the examples section. It should also be noted that the design method introduced above is quite flexible. No additional complexity is introduced when a P_o is desired off the negative real axis. $1/P_o$ is simply designated at the desired point and the $G_c^*(j\omega)$ line is rotated the required number of degrees.

III. EXAMPLES

To illustrate how the preceding results might be used, the following examples are given.

Example 1. It is desired to compensate the system shown in Figure 11 so that the output signal will be asymptotically stable of degree 0.1.

To simply guarantee output asymptoticity with the specified type of nonlinearity, the minimum Popov sector must be $[0,5]$. However, for the system to be asymptotically stable of degree 0.1, the Popov line (for some q) must intersect the negative real axis at a value ≤ -0.2 and must not be closer than 0.1 horizontal units to the $G^*(j\omega)$ curve at the closest point.

The $G(j\omega)$ and $G^*(j\omega)$ curves are shown in Figures 12 and 13. A lag-type compensator and an $w_c = 1.0$ are selected. To avoid the possibility of a borderline design, a P_o of -0.08 is used, as opposed to 0.1. The graphical design of the compensator is shown in Figure 14. The $1/P_o$ point is at $1/-0.08=-12.5$. To locate point $P_1 = 1/G^*(j1)$, note on Figure 13 that $G^*(j1) = -0.4-j0.2$ from which $1/G^*(j1) = -2+j1$ as shown. The parameter values, τ_1 and τ_2, are computed using the slope of the $G_c^*(j\omega)/P_o$ plot (see Figure 14) and Equation (25). The resulting compensator is given by

$$G_c(s) = \frac{1 + 1.59s}{1 + 10.5s} \quad . \tag{32}$$

Using a minimum number of points the $G^*(j\omega)G_c^*(j\omega)$ locus (see Figure 15) is easily sketched. Then, using (21), with (30) as a check,

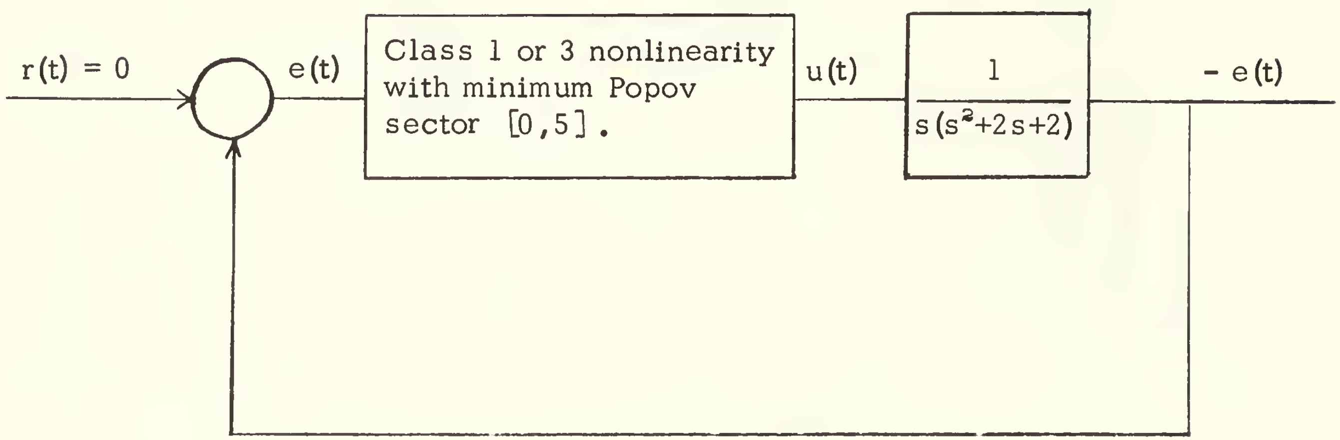

Fig. 11. System of Example 1.

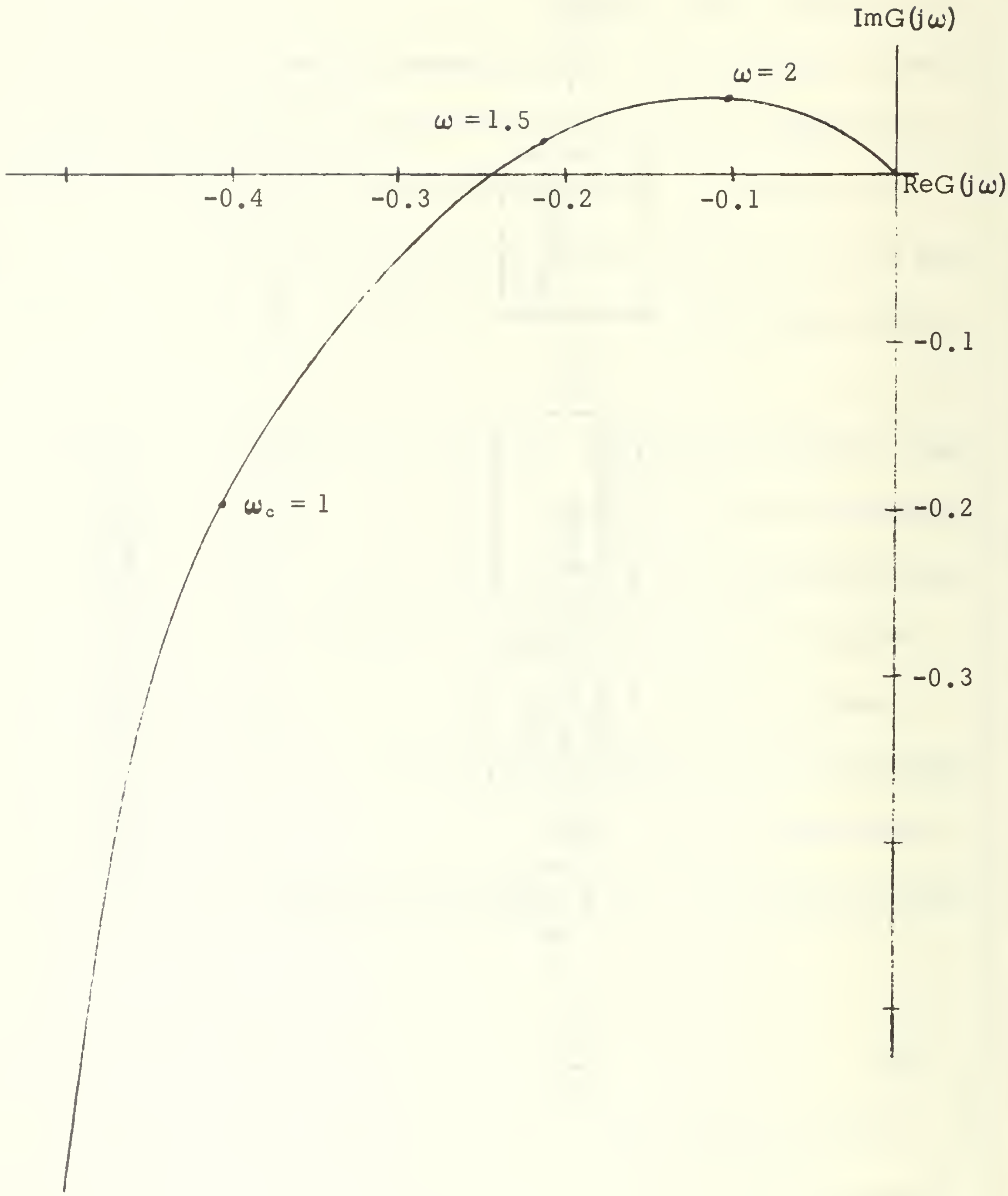

Fig. 12. G(jω) locus for Example 1.

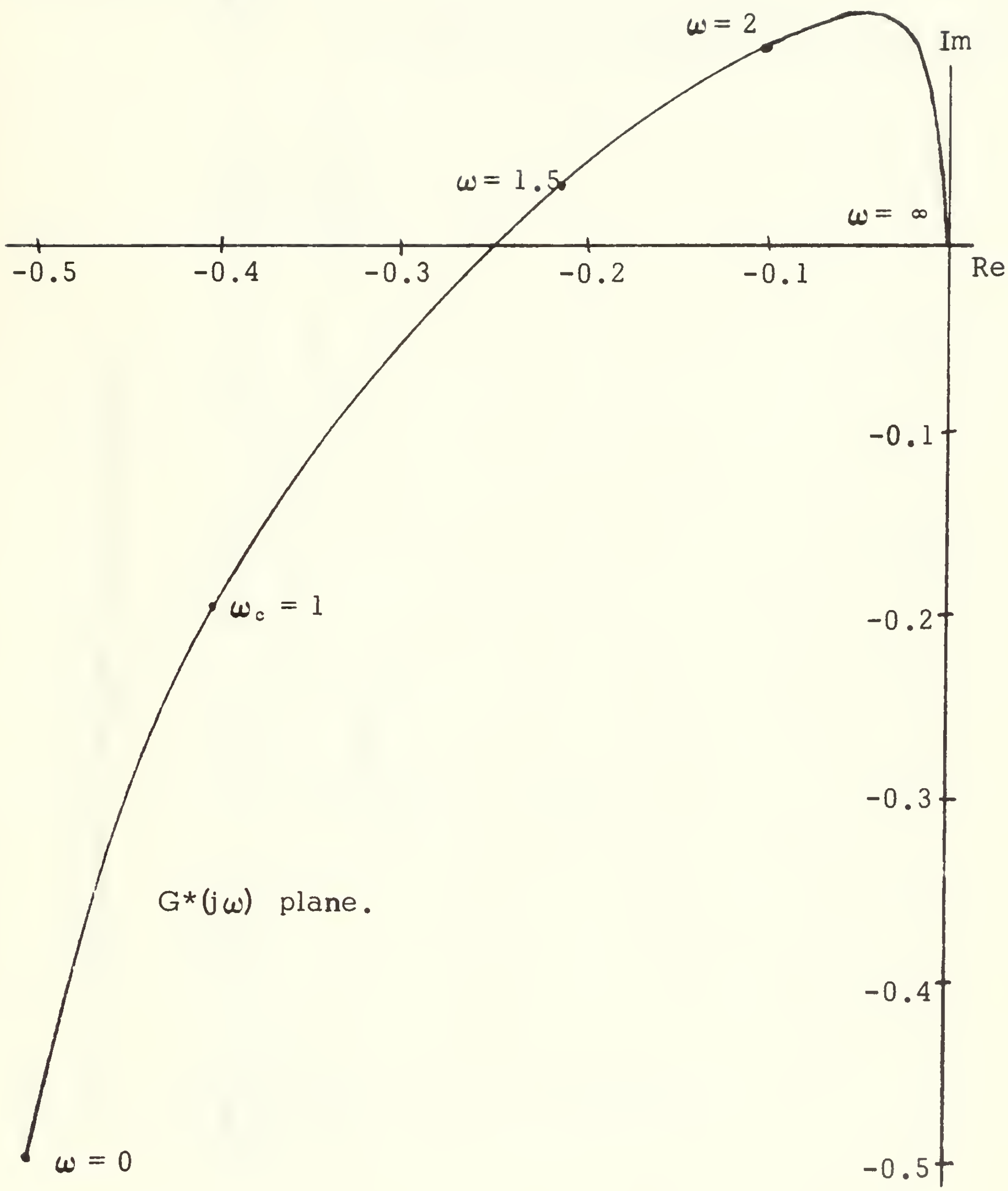

Fig. 13. $G^*(j\omega)$ locus for Example 1.

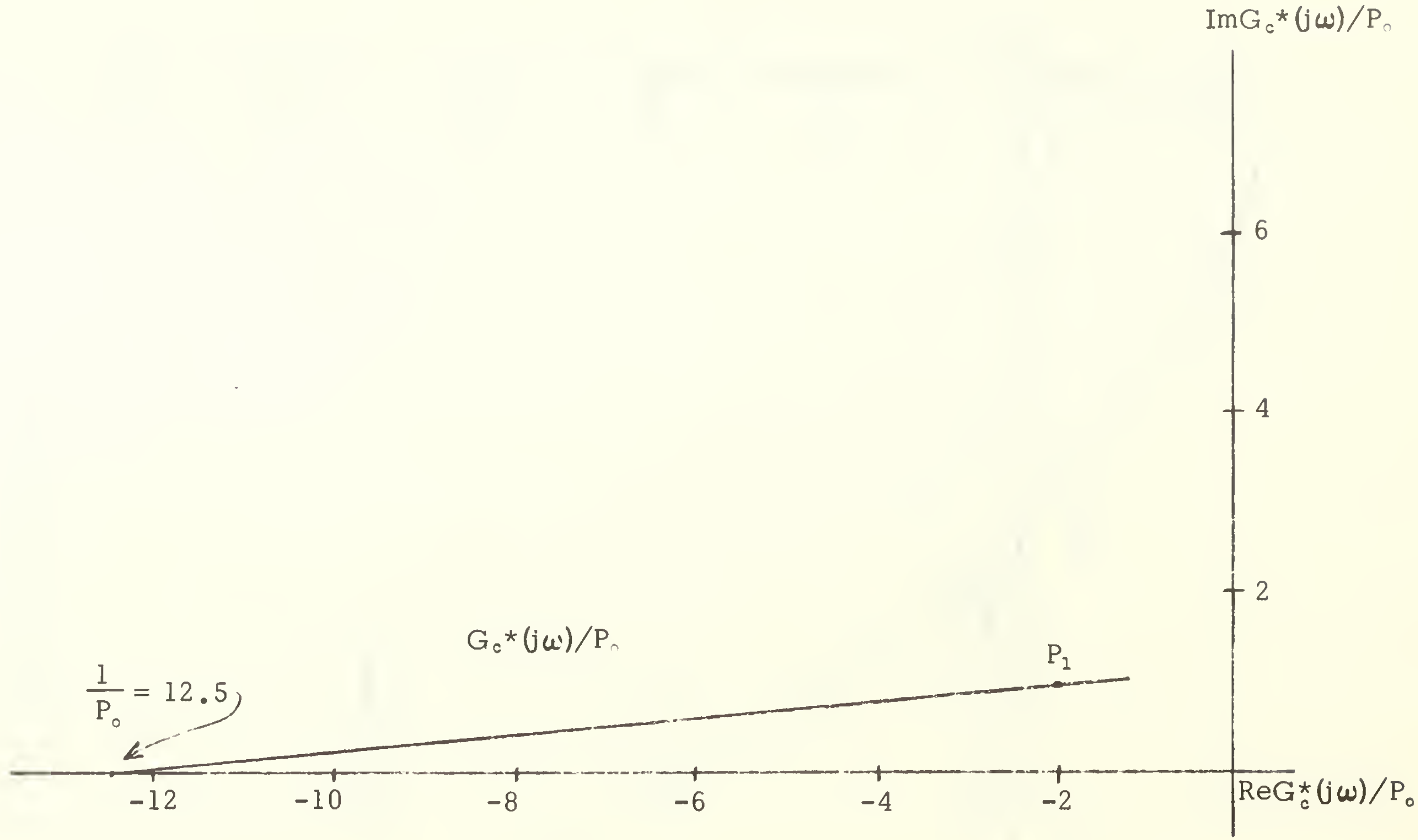

Fig. 14. Lag compensation design plot for Example 1.

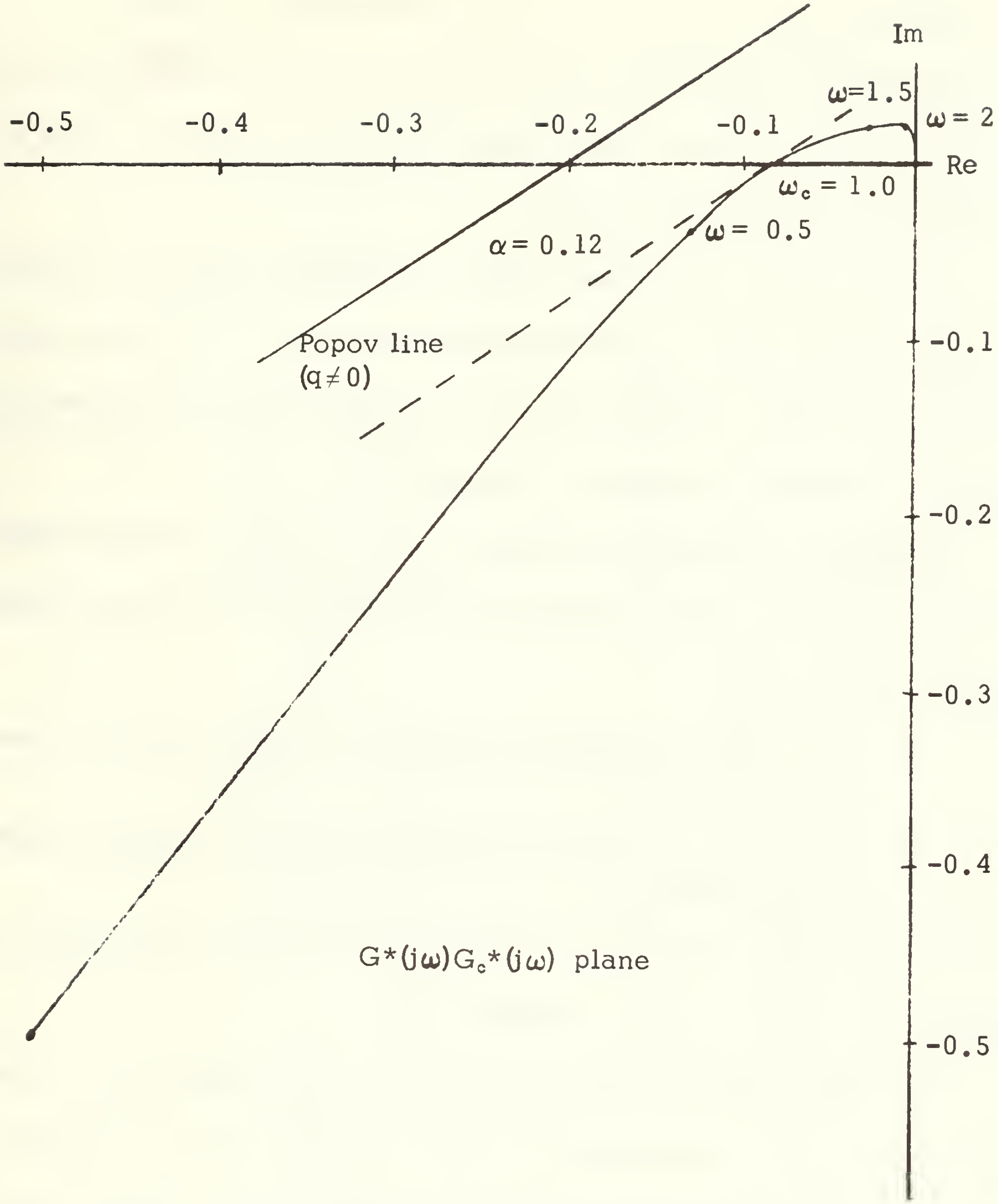

Fig. 15. $G^*(j\omega)G_c^*(j\omega)$ locus for Example 1.

the $[G(j\omega)G_c(j\omega)]^*$ locus can be quickly determined. In (21), the $(\omega^2-1)\mathrm{Im}G(j\omega)\mathrm{Im}G_c(j\omega)$ term can be evaluated by simply picking the values of $\mathrm{Im}G(j\omega)$ off the Nyquist plot of $G(j\omega)$ and using the expression for Y in (25) divided by ω for $\mathrm{Im}\ G_c(j\omega)$.

In this problem, the $G^*(j\omega)G_c^*(j\omega)$ and the $[G(j\omega)G_c(j\omega)]^*$ loci have the same negative real axis crossover at $\omega_c = 1$ and are virtually identical at other points. The Popov line is as shown in Figure 15 guaranteeing control and output asymptoticity of degree 0.12.

<u>Example 2</u>. Consider the nonlinear control system with time delay shown in Figure 16. Hsu and Meyer [2] studied this system and obtained the following results:

1. For a class 2 or class 4 nonlinearity ($q = 0$), the Popov criterion is satisfied for $0 < k \leq 1.85$ (see Figure 17).

2. For a class 1 or class 3 nonlinearity ($q \neq 0$), the Popov criterion is satisfied for $0 < k \leq 1.98$ (see Figure 17).

Now it is desired to compensate this system to permit a Popov sector of $[0,5]$ for a class 1 or class 3 nonlinearity.

The $G^*(j\omega)$ locus is shown in Figure 17. Lag compensation is chosen with an $\omega_c = 1.8$. Equation (30) quickly shows that if a P_0 is chosen which gives a minimum Popov sector, the initial design will not produce an unstable system but will actually give a greater degree of stability. Hence a P_0 of -0.2 is used. Designing the compensator as before (see Figure 18) gives the result

$$G_c(s) = \frac{1 + 1.27s}{1 + 4.7s} \tag{33}$$

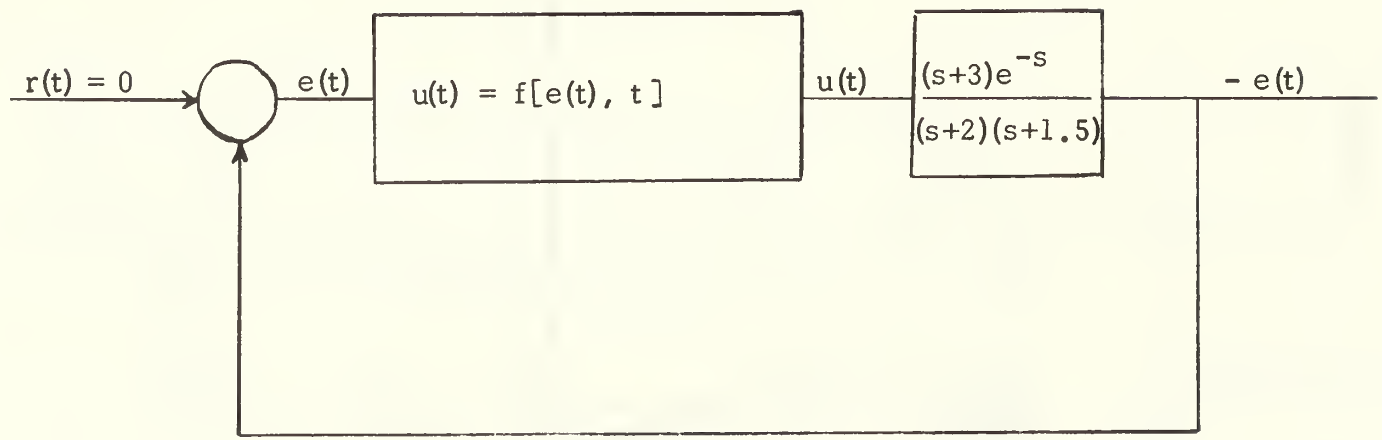

Fig. 16. System for Example 2.

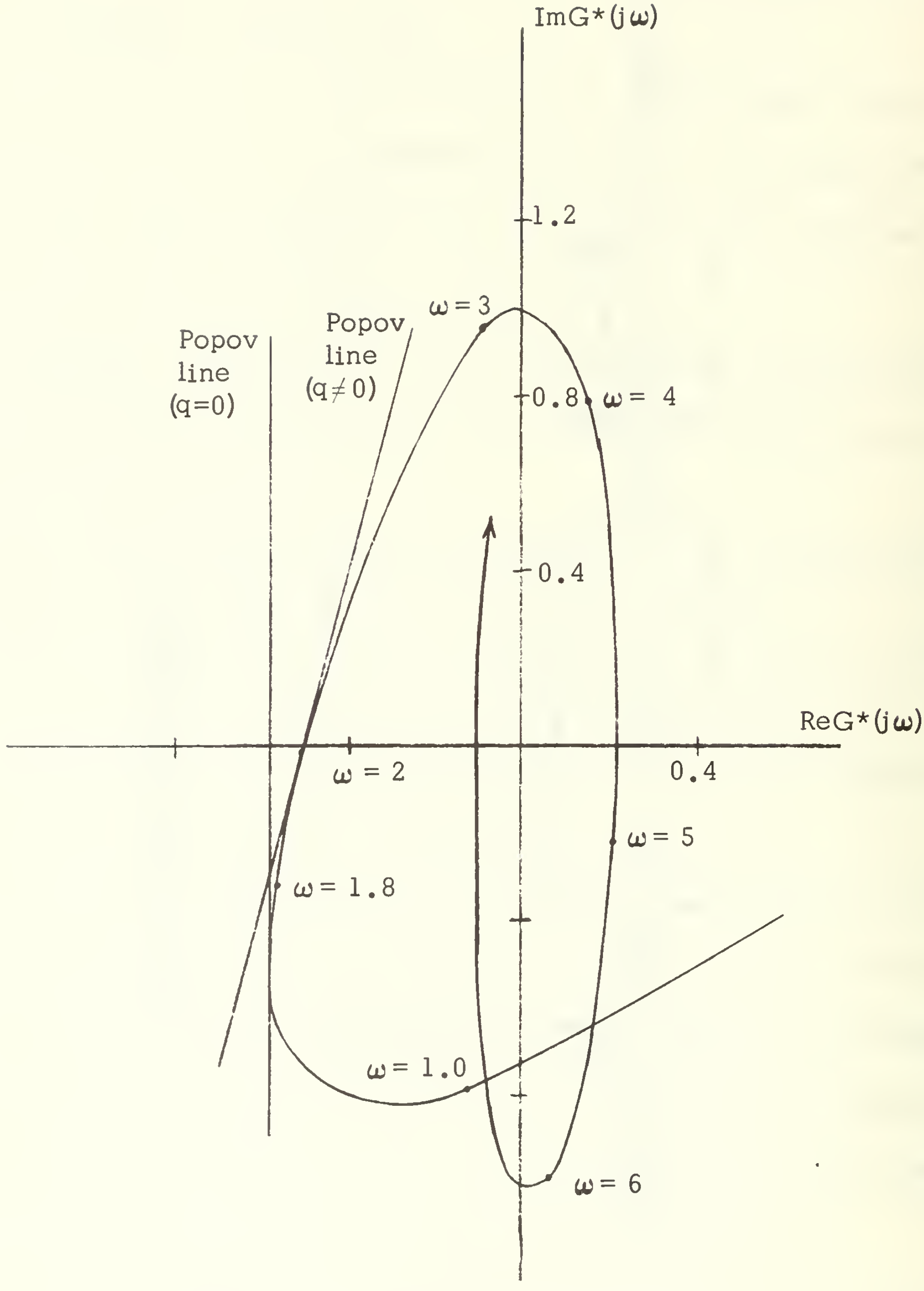

Fig. 17. G*(jω) locus for Example 2.

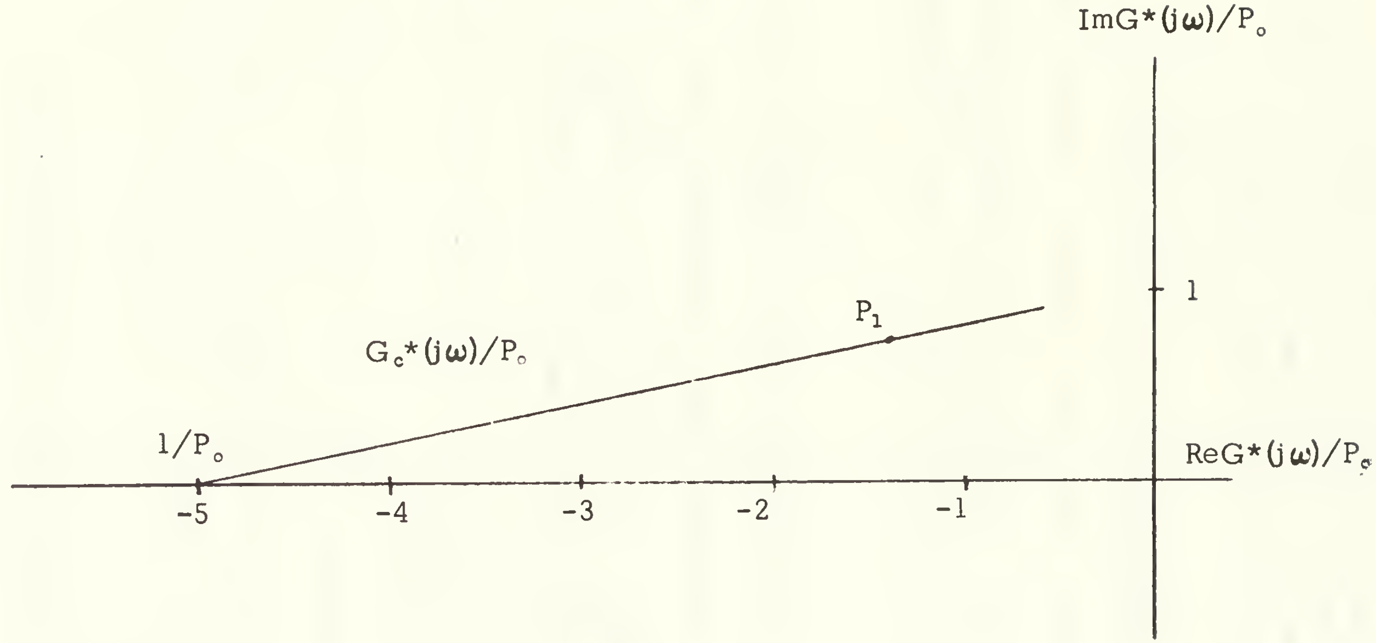

Fig. 18. Lag compensation design plot for Example 2.

Therefore:

1. For a class 2 or class 4 nonlinearity ($q = 0$), the Popov criterion is satisfied for $0 < k \le 4.45$ (see Figure 19). This example illustrates, as a sidelight, how easily the design method may also be applied to the case when $q = 0$, although the advantages are not quite as great as when $q \neq 0$.

2. For a class 1 or class 3 nonlinearity ($q \neq 0$), the Popov criterion is satisfied for $0 < k \le 5$ (see Figure 19.)

The permissible Popov sector will actually be slightly greater than above as indicated by (30). If exact sector limitations are desired, these are quickly obtained by using (21) and (30) to obtain the $[G(j\omega)G_c(j\omega)]^*$ locus.

Example 3. Consider the actuator or indirect control system of Figure 20. Hsu and Meyer [2] also studied this sytem using the generalized Popov theorem. Their intent was to determine the critical feedback gain $F = F_c$ such that for feedback gains $F \ge F_c$, the output signal $y(t)$ would asymptotically approach a steady-state value dependent on the constant input value, r.

The modified block diagram of this system is shown in Figure 21. This zero-input model is possible because the constant input, r, can be combined with the constant-initial-condition-response term with no loss of generality.

Using the modified frequency response function,

$$G^*(j\omega) = G_2^*(j\omega) - jF \tag{34}$$

where

$$G_2^*(j\omega) = -\frac{24 + \omega^2}{36 + 13\omega^2 + \omega^4} - j\,\frac{36 - \omega^2}{36 + 13\omega^2 + \omega^4} \; . \tag{35}$$

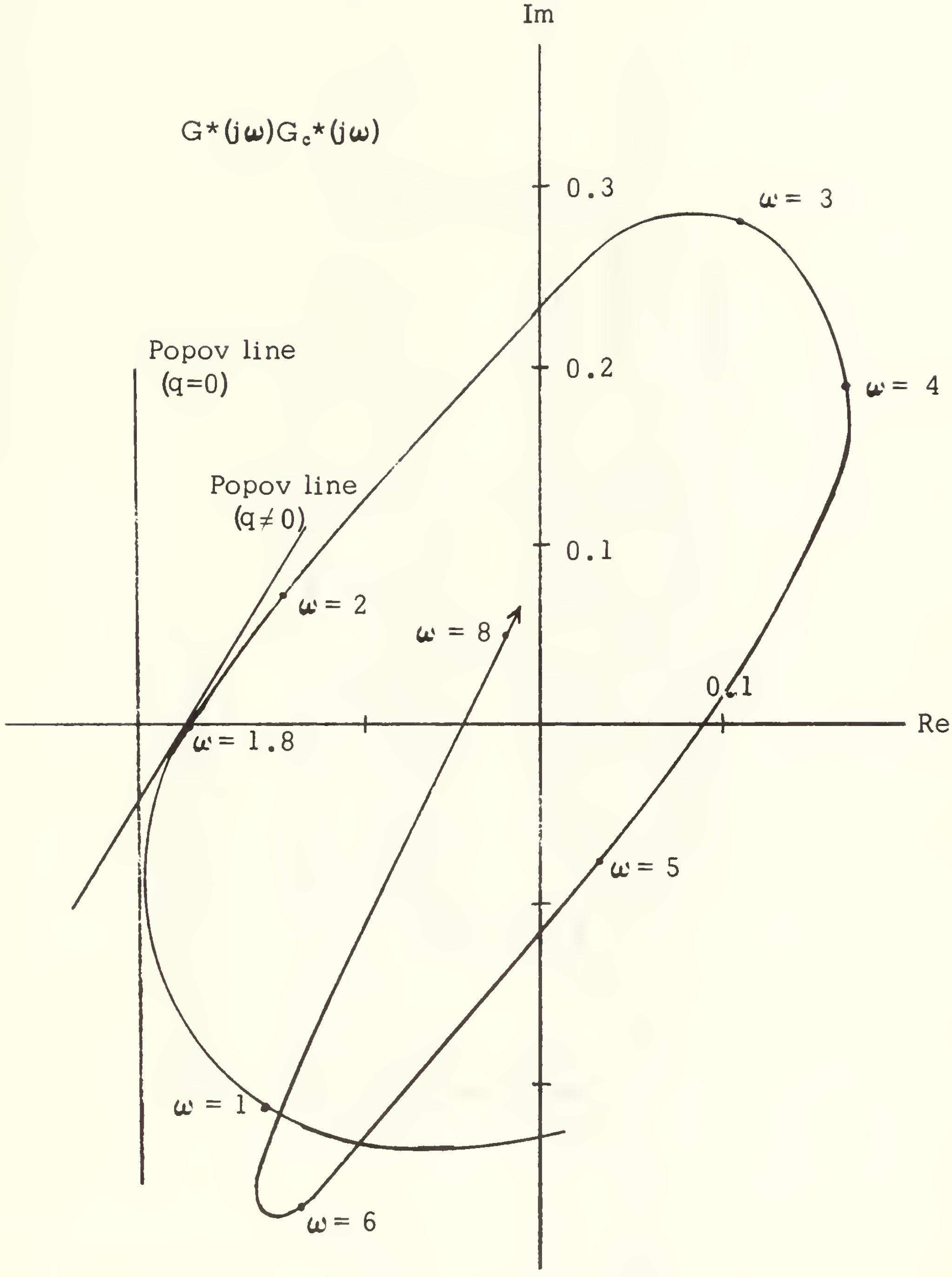

Fig. 19. $G^*(j\omega)G_c^*(j\omega)$ locus for Example 2.

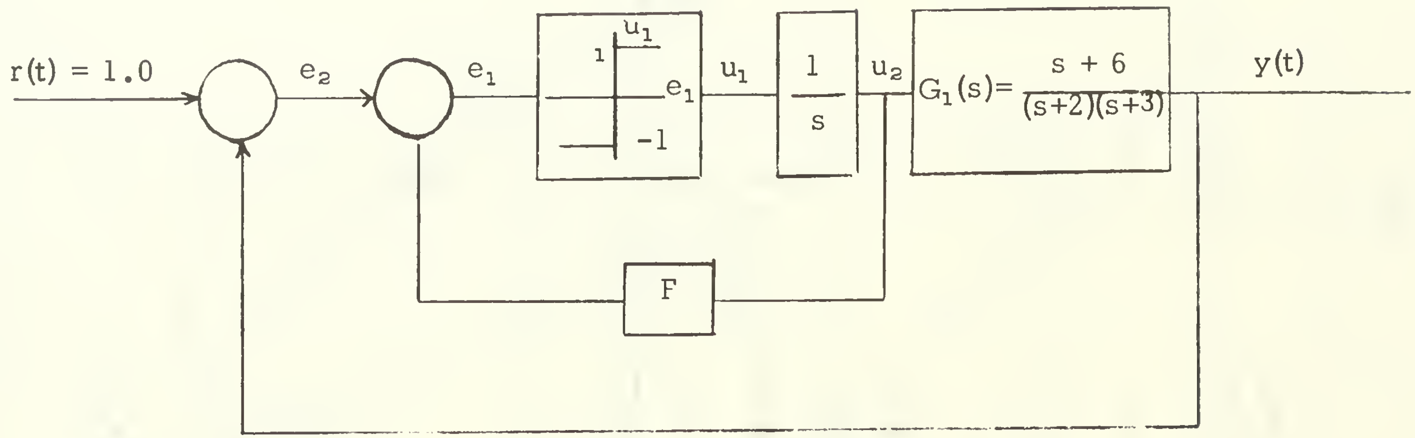

Fig. 20. System for Example 3.

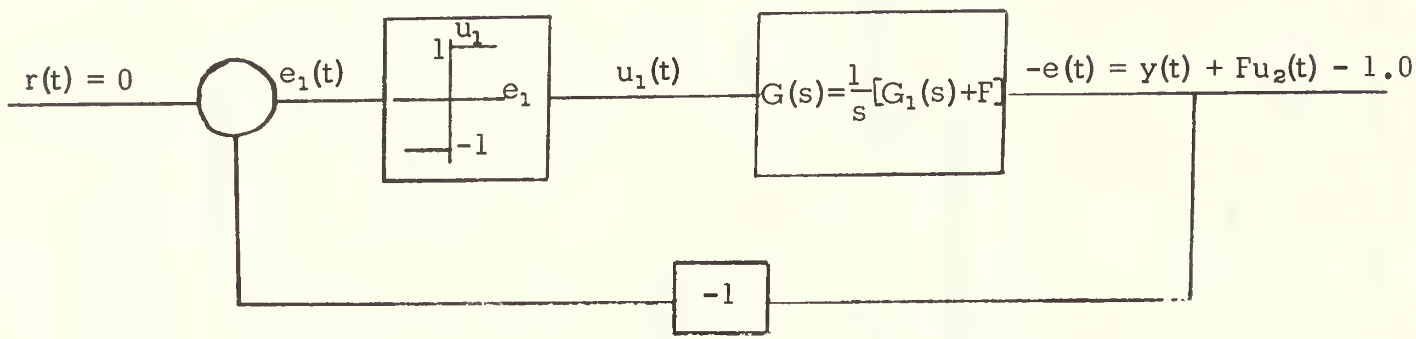

Fig. 21. Modified block diagram for system of Example 3.

For asymptotic stability, the Popov criterion with $k = \infty$ then requires that

$$\mathrm{Re}G^*(j\omega) - q\,\mathrm{Im}G^*(j\omega) = \mathrm{Re}G_2^*(j\omega) - q(\mathrm{Im}G_2^*(j\omega) - F) > 0.$$

$G_2^*(j\omega)$ is plotted in Figure 22. Since the $\mathrm{Im}G_2^*(j\omega)$ has a maximum value of approximately 0.006, the Popov criterion is satisfied with some $q > 0$ if $F > 0.006$ and the system is then control and output asymptotic. Hence $e(\infty) = 0$ and

$$y(\infty) = \frac{G_1(0)}{F + G_1(0)} = \frac{1}{F + 1}\, r \tag{36}$$

for this system. Therefore the range of F for which the system is control and output asymptotic is extremely important since F directly affects the steady-state error.

Digital simulation of the system, with $F = 0.006$, provided results agreeing with the theoretical values. The response of the simulated system to an input value of $r = 1.0$ is shown in Figure 23.

Assume now that design requirements specify a steady-state error which requires the system of Figure 20 to be asymptotically stable for $F = 0.004$. Hence the system must be compensated so that the maximum value of the $\mathrm{Im}G_2^*(j\omega)$ is less than 0.004. For comparison purposes, the unit step response of the uncompensated system with $F = 0.004$ is shown in Figure 24. Lead compensation, $\omega_c = 8.8$, and $P_o = -.017$ are chosen. The compensator design is shown in Figure 25 with the result

$$G_c(s) = \frac{1 + 0.1s}{1 + 0.079s} \tag{37}$$

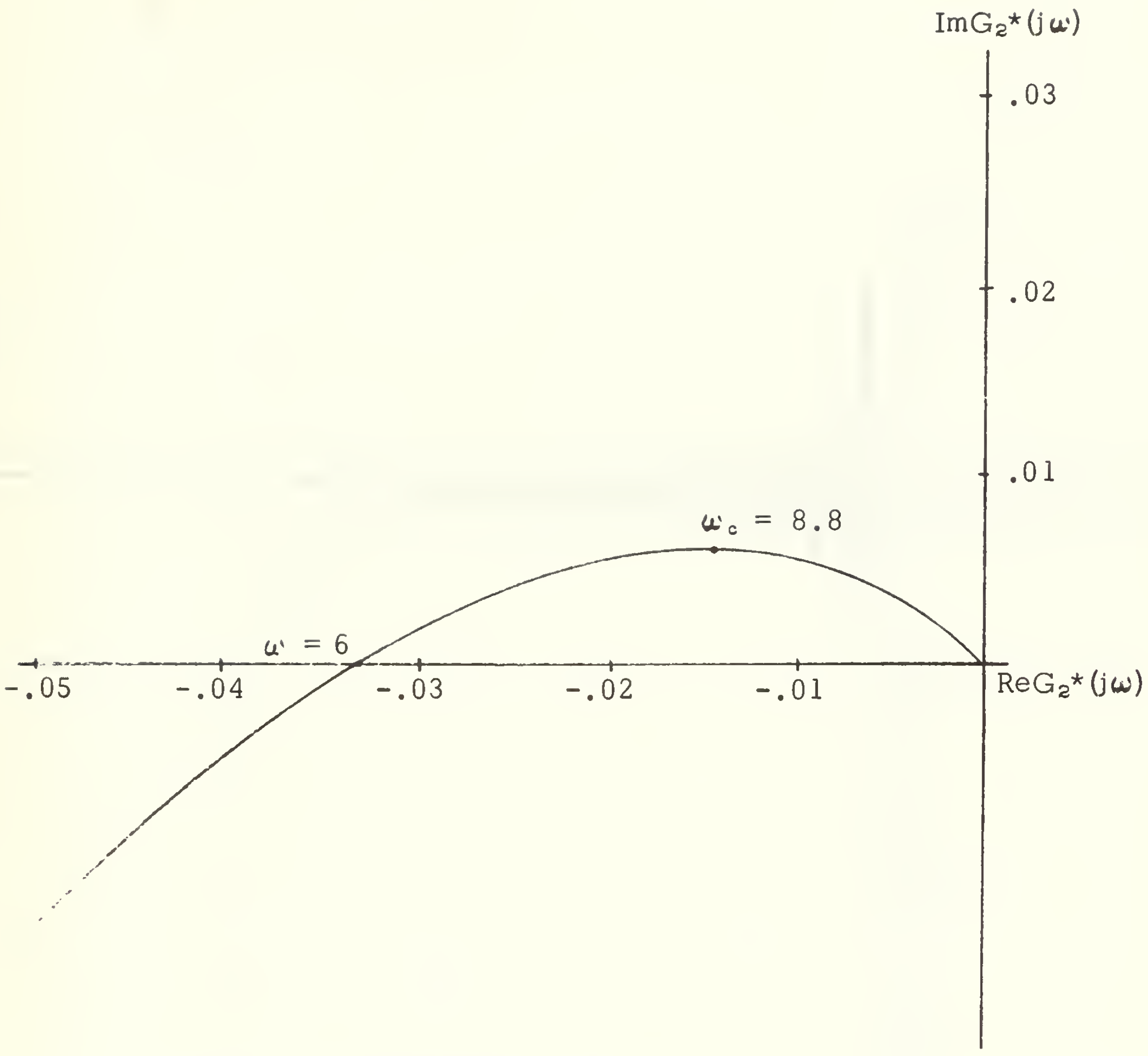

Fig. 22. $G_2^*(j\omega)$ locus for Example 3.

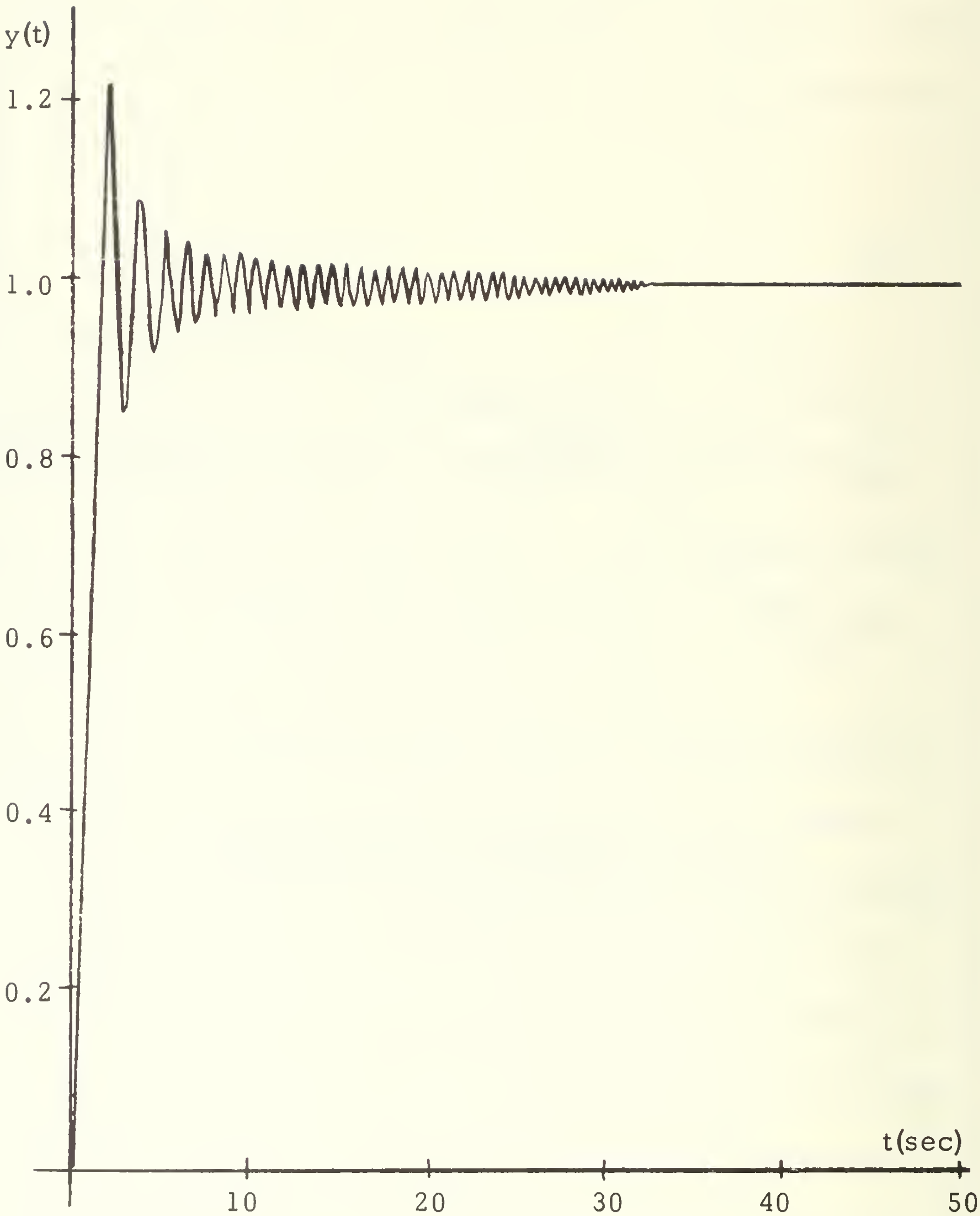

Fig. 23. Response of the system of Figure 20 to a unit step input. F = 0.006.

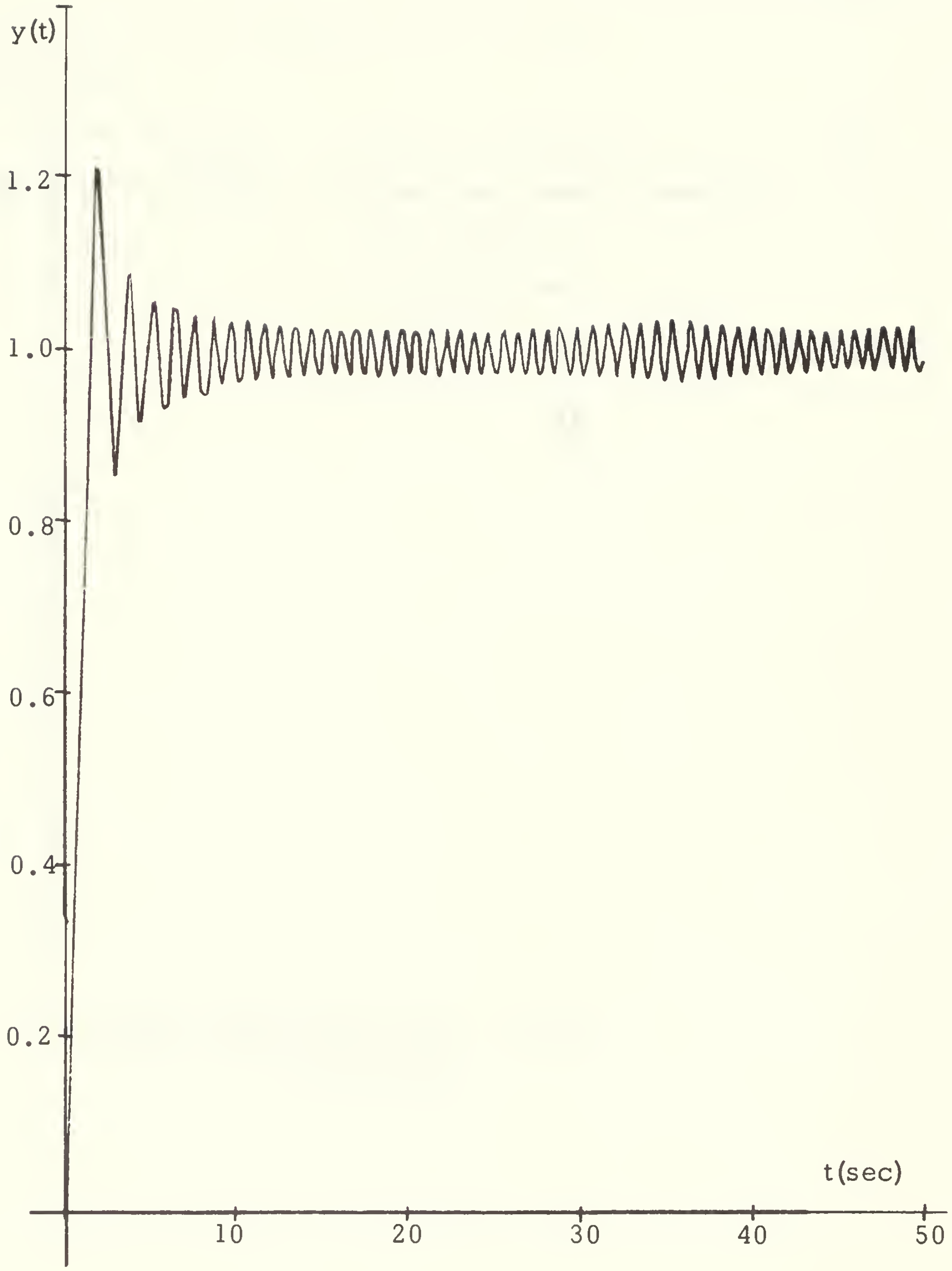

Fig. 24. Response of the system of Figure 20 to a unit step input. F = 0.004.

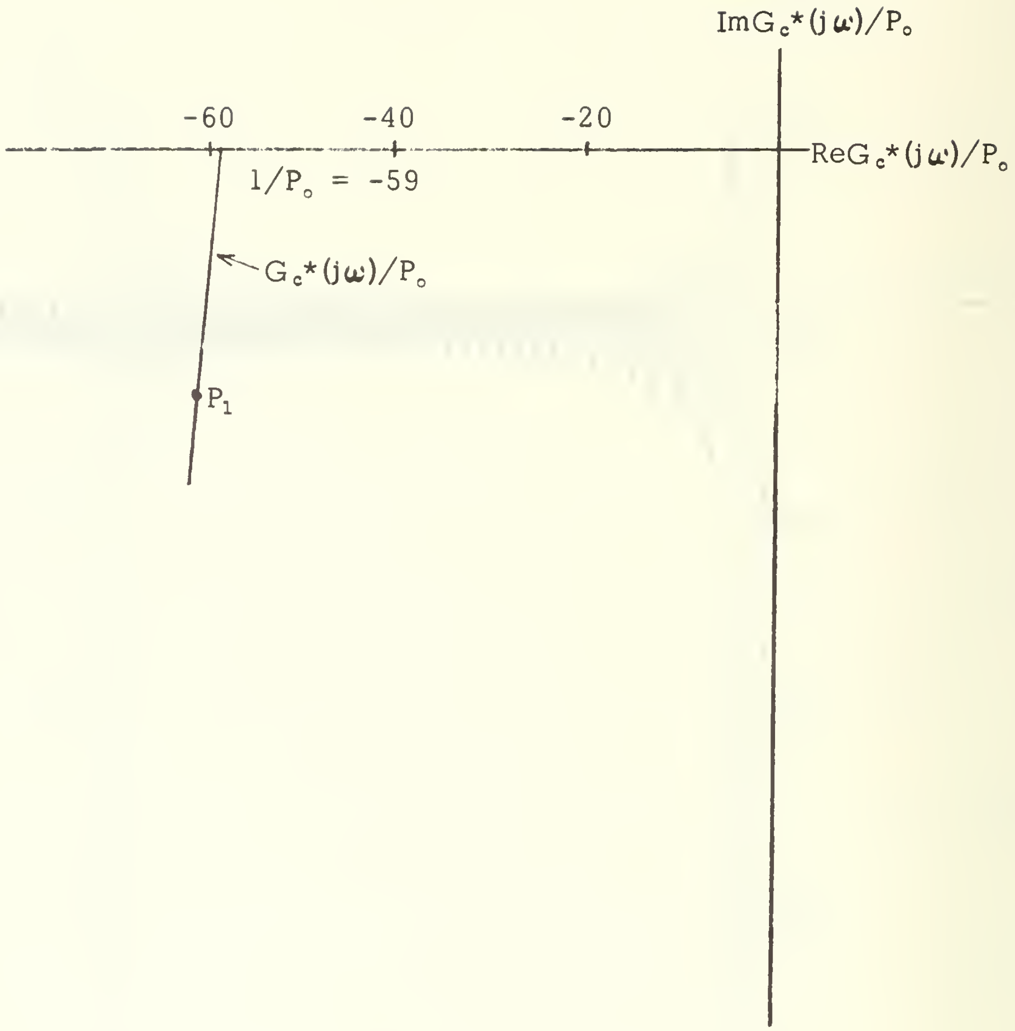

Fig. 25. Lead compensation design plot for Example 3.

The $G_c^*(j\omega)G_2^*(j\omega)$ locus is shown in Figure 26.

It should be noted that, in this case, the T term of (29) is of no consequence since it is only real and will have no effect on the maximum imaginary value of the compensated curve. Hence

$$\max \operatorname{Im} [G_c(j\omega)G_2(j\omega)]^* = \max \operatorname{Im} G_c^*(j\omega)G_2^*(j\omega) \qquad (38)$$

and the design is completed. The response of the compensated system with F = 0.004 and r = 1.0 is shown in Figure 27.

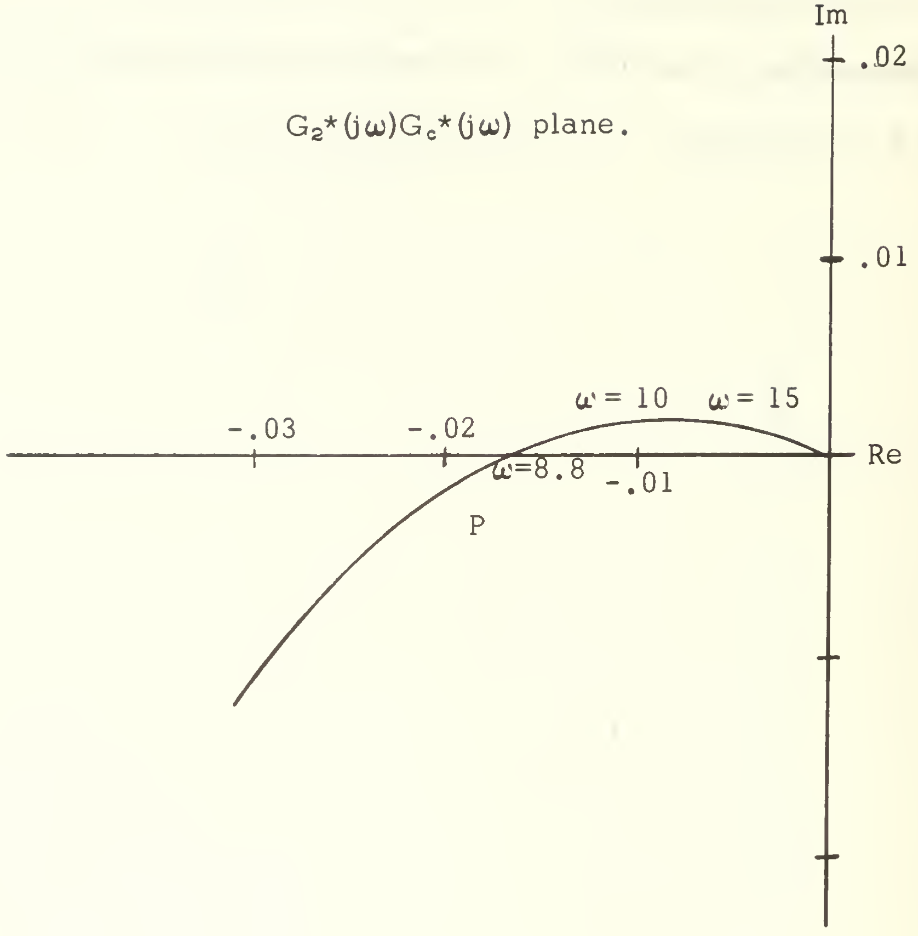

Fig. 26. $G_2^*(j\omega)G_c^*(j\omega)$ locus for Example 3.

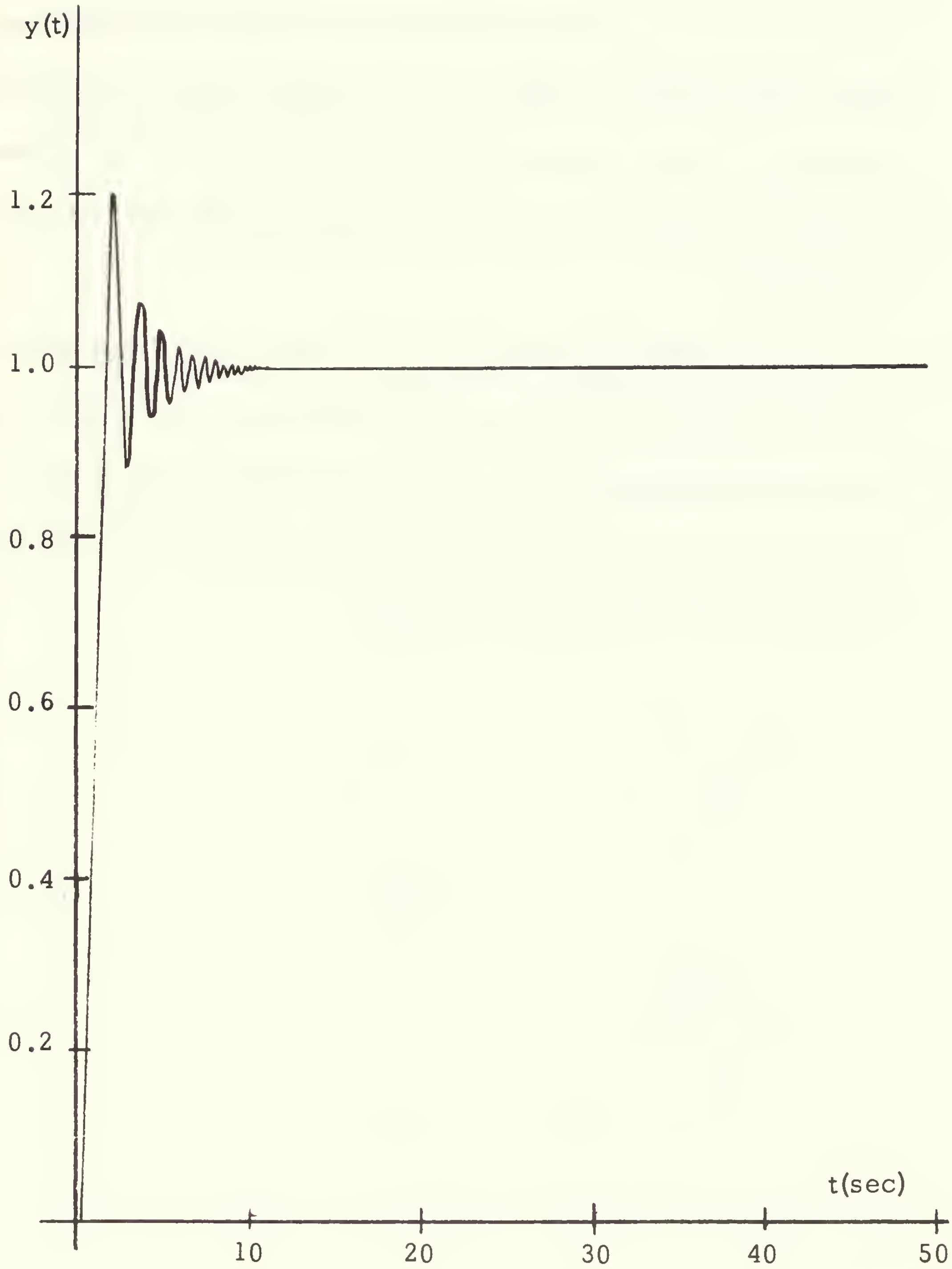

Fig. 27. Response of the compensated system of Example 3 to a unit step input. F = 0.004.

IV. CONCLUSION

A design method, based on the Popov stability criterion, has been developed for the large class of nonlinear systems which can be modeled in the configuration of Figure 1. The design procedure uses the modified frequency-response polar loci for the linear portion of the system. This permits the engineer to utilize the clear physical picture inherent in the Nyquist plot.

As a major step in the development of the design procedure, it was shown that the modified frequency-response polar plots for lag and lead compensation networks are straight lines. This provided the basis for the development of a rapid, straightforward design method which could be quite useful in engineering practice.

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13. ABSTRACT

A method for nonlinear system design, based on the generalized Popov stability criterion, is developed. It is shown that, although the design method is applicable for all values of q in the Popov theory, it is particularly useful when q is non-zero, since it greatly simplifies the design effort for this case. All designs are accomplished in conjunction with the Nyquist and modified Nyquist loci. Basic to the design procedure is the development and utilization of the modified frequency-response polar loci for lag and lead compensation networks. Three examples, one with digital simulation, are included to illustrate the procedure described.

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